Differential sum rule for the relaxation rate in dirty superconductors

Andrey V. Chubukov,[1] Ar. Abanov,[2] and D. N. Basov[3]
[1]*Department of Physics, University of Wisconsin, Madison, Wisconsin 53706, USA*
[2]*Los Alamos National Laboratory, MS 262B, Los Alamos, New Mexico 87545, USA*
[3]*Department of Physics, University of California, San Diego, La Jolla, California 92093, USA*
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We consider the differential sum rule for the effective scattering rate $1/\tau(\omega)$ and optical conductivity $\sigma_1(\omega)$ in a dirty BCS superconductor, for arbitrary ratio of the superconducting gap Δ and the normal state constant damping rate $1/\tau$. We show that if τ is independent of *T*, the area under $1/\tau(\omega)$ does not change between the normal and the superconducting states, i.e., there exists an exact differential sum rule for the scattering rate. For *any* value of the dimensionless parameter $\Delta\tau$, the sum rule is exhausted at frequencies controlled by Δ. We show that in the dirty limit the convergence of the differential sum rule for the scattering rate is much faster then the convergence of the *f*-sum rule, but slower then the convergence of the differential sum rule for conductivity.

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Optical constants of solids follow a variety of sum rules.[1] The origins of the sum rules can be traced back to fundamental conservation laws and are intimately connected to the causality of the electromagnetic response leading to Kramers-Kronig relationships between the real and imaginary parts of the optical constants. The analysis of the sum rules is a powerful tool to study the distribution of the spectral weight in correlated electron systems.

The dynamics of conducting carriers is usually described in terms of the effective scattering rate $1/\tau(\omega)$ and the effective mass $m^*(\omega)$[2]

$$\tau^{-1}(\omega)=\frac{\omega_{pl}^2}{4\pi}\mathrm{Re}\left[\frac{1}{\sigma(\omega)}\right],\frac{m^*}{m_b}=-\frac{\omega_{\mathrm{pl}}^2}{4\pi}\frac{1}{\omega}\mathrm{Im}\left[\frac{1}{\sigma(\omega)}\right], \quad (1)$$

where $\sigma(\omega)=\sigma_1(\omega)+i\sigma_2(\omega)$ is the complex optical conductivity, $\omega_{\mathrm{pl}}^2=4\pi ne^2/m_b$ is the plasma frequency, and m_b is the band mass.[2] Under special circumstances (e.g., when the Eliashberg theory is valid), the spectra of $1/\tau(\omega)$ and m^* can be linked to the real and imaginary parts of the electronic self energy.

The conductivity, that characterizes the absorbing power of a solid, obeys the *f*-sum rule $\int_0^\infty d\omega\sigma_1(\omega)=\omega_{\mathrm{pl}}^2/8$. This sum rule reflects the conservation of the number of particles and physically implies that at a given particle density, the total absorbing power does not depend on the details of the interactions, and is determined only by the total number of particles in the system.[1] It is particularly relevant for a superconductor where the conductivity acquires a δ-functional piece proportional to the superfluid density n_s, and the sum rule transforms into $\int_{+0}^\infty d\omega\sigma_1(\omega)=\omega_{\mathrm{pl}}^2/8(1-n_s/n)$. As n_s/n can be measured independently, the sum rule is a valuable tool to study the transformation of the spectral weight associated with the superconductivity.

The issue raised in series of recent works[3–6] is whether there exists any similar sum rule for the relaxation rate. At the first glance, the answer is negative as the integral over $1/\tau(\omega)$ diverges and hence has no physical meaning. This can be readily illustrated with a simple Drude model where $\sigma(\omega)=(\omega_{\mathrm{pl}}^2/4\pi)/(1/\tau-i\omega)$, i.e., $\tau(\omega)=\tau$, and $\int_0^\infty d\omega\tau^{-1}(\omega)$ is infinite. This implies that there is no sum rule for $1/\tau(\omega)$ that could be traced to a conservation law. However, one still can argue that the frequency integral of the scattering rate has some physical meaning. Basov *et al.*[3] argued, in connection with the cuprates, that for experimentally relevant frequencies $1/\tau(\omega)$ is numerically close to an effective scattering rate $1/\tau_{\mathrm{SR}}$ (defined below) that is related to the dielectric function and obeys a physically motivated sum rule. In this situation, one can argue that there should exist an approximate sum rule if the frequency integral over $1/\tau(\omega)$ is taken in finite limits. This approach was thoroughly analyzed by Marsiglio *et al.*[4]

In this paper, we analyze whether there exists an *exact* sum rule for the full frequency integral of the difference between $1/\tau(\omega)$ at two different temperatures in the normal state, and, what is actually more relevant, between $1/\tau(\omega)$ in the normal and superconducting states. For a Drude metal, the frequency integral over the difference of $1/\tau(\omega)$ at different *T* in the normal state is either infinite or vanishes depending on whether or not $1/\tau$ depends on *T*. We argue that zero and infinity are the only two options for the differential integral in any model, and that if $\tau(\omega)$ is *T* independent, the differential sum rule for the relaxation rate is *exact* between the normal state and a BCS superconductor.

We also analyze what controls the rate of convergence of the sum rules for conductivity and $1/\tau$. This issue is relevant for experimental applications as when sum rules are applied to the analysis of the experimental data, one always encounters a problem that actual data is available over limited frequency intervals. It is therefore imperative to know rapidly sum rules converge to estimate the accuracy of the experiment-based sum rule analysis.

The analysis in the clean limit $\Delta\tau\gg1$, where Δ is the superconducting gap, and τ is independent on *T* was performed earlier,[5] as a by-product of the modeling for the cuprates. In this limit, Δ is the largest energy scale in the problem, and the differential sum rule between the normal state and a BCS superconductor is exact and is exhausted at

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frequencies of order Δ. Here we consider arbitrary $\Delta\tau$. We show that even in the dirty limit $\Delta\tau \ll 1$, the differential sum rule is exact and is furthermore still exhausted at frequencies of order Δ. This result is not intuitively obvious as the conductivity sum rule in the dirty limit is exhausted only at $\omega \sim 1/\tau \gg \Delta$. We also found that the functional form of $1/\tau_{sc}(\omega)$ only weakly depends on $\Delta\tau$ at $\omega > 2\Delta$. Here and below we use subscripts "sc" and "n" for superconducting and normal states, respectively

The fact that the differential sum rule for $1/\tau$ is exhausted at $\omega \propto \Delta$ in the dirty limit may be relevant for the interpretation of the data from some high-T_c materials. The frequency below which fermionic excitations become incoherent may arise from the energy scale $\omega \sim 1/\tau$. The convergence of the differential sum rule at $O(\Delta)$ and the near independence of $\tau_{sc}(\omega)$ on $\Delta\tau$ implies that as long as the system possesses a sharp superconducting gap the differential sum rule for $1/\tau$ is not affected by the increased fermionic incoherence. Indeed in high-T_c superconductors, $1/\tau$ is temperature dependent. The issue of whether or not the differential sum rule is exact between the normal and superconducting states could then only be experimentally addressed if measurements could be done in both states at the same T. This is obviously impossible. What can be clearly observed, at least in YBCO, is the redistribution of the spectral weight between the normal and superconducting states. If this redistribution is over scales of $O(\Delta)$ then the frequency integral can be restricted to a few times Δ. The approximate sum rule can then be analyzed in the hope that the T dependence of the relaxation rate is a minor effect compared to the huge change in $1\tau(\omega)$ which takes place between the normal and superconducting states. Note in this regard that since $1/\tau$ is expressed via both $\mathrm{Re}\,\sigma(\omega)$ and $\mathrm{Im}\,\sigma(\omega)$, the analysis of the differential sum rule for $1/\tau$, even though it is only approximate, still yields information about the spectral weight distribution in a superconductor, which is complimentary to the information from the f-sum rule that involves only $\mathrm{Re}\,\sigma(\omega)$

The issue whether or not $1/\tau$ differential sum rule is satisfied in dirty BCS superconductors was a subject of recent controversy. Basov *et al.*[3] argued that in both clean and dirty limits, $1/\tau_{sc}(\omega)$ vanish below 2Δ, but overshoots the normal state $1/\tau$ at larger frequencies. In that paper, the profile of $1/\tau_{sc}(\omega)$ was visually related to that of the fermionic density of states, for which the sum rule is exact and reflects the conservation of the number of particles. Homes *et al.*[6] correctly observed that the relaxation rate and conductivity are expressed via a current-current correlator, and therefore scale with the joint density of states of two fermions about which, they conjectured, no rigorous statements can be made. They computed the frequency integral of $1/\tau_{sc}(\omega) - 1/\tau_n(\omega)$ numerically for a BCS superconductor $\tau_n(\omega) = \tau$ with $\tau\Delta = 1/2$ and found that even at ω as high as 15Δ, it is still about 25% of its maximum value at $\omega = 2\Delta$. They concluded, based on this numeric, that the value of sum rule integral depends on the choice for cutoff frequency, and that if the integral is truncated at $\omega = O(\Delta)$, a finite value is expected. Below we demonstrate explicitly that the differential sum rule is in fact exhausted at the energy scale controlled by Δ, however, a rather slow convergence found in Ref. 6 is real and is due to a weak $(\ln\omega)/\omega^2$ decay of $1/\tau_{sc}(\omega)$ at large frequencies.

We begin with the general argument that zero and infinity are the only two options for $\int_0^\infty d\omega/\tau(\omega)$. This conjecture can be verified by applying the Kubo formula that relates $\sigma(\omega)$ with the full retarded current-current correlator $\Pi(\omega)$: $\sigma(\omega) = (\omega_{\mathrm{pl}}^2/4\pi)\Pi(\omega)/(-i\omega+\delta)$. Substituting this relation into $1/\tau(\omega)$ we find that $1/\tau(\omega) = \mathrm{Im}\,S(\omega)/\omega$ where $S(\omega) = -\omega^2/\Pi(\omega)$. Both $\mathrm{Im}\,\Pi(\omega)$ and $S(\omega)$ are odd functions of frequency. Since $\Pi(\omega)$ is analytic in the upper half-plane of complex ω and does not have zeros (that can be checked explicitly, see below), $S(\omega)$ is also an analytic function in the upper half-plane. The analyticity implies that, by Kramers-Kronig relations

$$\int_0^\infty \frac{1}{\tau(\omega)} = \int_0^\infty \frac{S(\omega)}{\omega} d\omega = \frac{\pi}{2} \mathrm{Re}\,S(0) + C, \tag{2}$$

where $C=0$ if the integral converges, and $C=\infty$ if it does not. It is easy to show that $\mathrm{Re}\,S(0)$ vanishes both in the normal and in the superconducting state. In the normal state, $\mathrm{Re}\,\Pi(\omega) \propto \omega^2$, $\mathrm{Im}\,\Pi(\omega) \propto \omega$ at the lowest frequencies, hence $S(\omega) \propto \omega^2$ and $S(0)=0$. In the superconducting state $\mathrm{Re}\,\Pi(0)$ is finite, while $\mathrm{Im}\,\Pi(0)=0$, hence $S(\omega)$ again scales as ω^2, and again $S(0)=0$. Hence $\int d\omega/\tau(\omega) = C$, i.e., it is either zero or infinite. For nondifferential sum rule, $C = \infty$ as at large frequencies, $\Pi(\omega)$ tends to 1, hence $S(\omega) \approx \omega^2$, and the integral in Eq. (2) diverges. This confirms that there is no physically motivated sum rule for $1/\tau(\omega)$.

There are two ways to improve the situation. First, one can introduce an effective $S_{\mathrm{eff}}(\omega)$ that converges at high frequencies and is close to $S(\omega)$ at experimentally relevant frequencies. Then one can hope to obtain an experimentally meaningful approximate sum rule. For the scattering rate, the natural choice, suggested by Basov *et al.*,[3] is to introduce an effective scattering rate

$$\frac{1}{\tau_{\mathrm{eff}}} = \frac{\omega_{\mathrm{pl}}^2}{\omega} \mathrm{Im}\left[1 - \frac{1}{\epsilon(\omega)}\right], \tag{3}$$

where $\epsilon(\omega) = 1 + 4\pi i \sigma(\omega)/\omega$ is the dielectric function. For this scattering rate, $S_{\mathrm{eff}}(\omega) = 1 - \omega/[\omega + 4\pi i \sigma(\omega)]$ is an analytic function, and at high frequencies it vanishes as $S_{\mathrm{eff}}(\omega) = O(1/\omega^2)$. The Kramers-Kronig relations are then applicable, and

$$\int_0^\infty \frac{d\omega}{\tau_{\mathrm{eff}}(\omega)} = \frac{\pi\omega_{\mathrm{pl}}^2}{2} \mathrm{Re}\left[1 - \frac{1}{\epsilon(\omega)}\right]_{\omega=0} = \frac{\pi}{2}\omega_{\mathrm{pl}}^2. \tag{4}$$

For a Drude metal,

$$\frac{1}{\tau_{\mathrm{eff}}(\omega)} = \frac{1}{\tau} \frac{\omega_{\mathrm{pl}}^4}{(\omega^2 - \omega_{\mathrm{pl}}^2)^2 + \omega^2/\tau^2}. \tag{5}$$

At frequencies $\omega \ll \omega_{\mathrm{pl}}$, the correction term in the denominator can be neglected, and $1/\tau_{\mathrm{eff}}(\omega) \approx 1/\tau$. As the plasma frequency can well be larger than the fermionic bandwidth, the actual integration of experimentally measured $1/\tau(\omega)$ may not extend to $\omega\sim\omega_{\rm pl}$, i.e., in the measured range $1/\tau_{\rm eff}(\omega)$ and $1/\tau(\omega)$ are nearly the same. Still, however, this does not imply that there is an "approximate" sum rule for $1/\tau$ as $\int d\omega/\tau(\omega)$ and $\int d\omega/\tau_{\rm eff}(\omega)$ both diverge when the integration is restricted to $\omega\ll\omega_{\rm pl}$. Only when the frequency integration extends to $\omega>\omega_{\rm pl}$, $\int_0^\omega dx/\tau_{\rm eff}(x)$ converges to the sum rule value. In other words, typical frequencies for $\int_0^\infty 1/\tau_{\rm eff}(\omega)$ are of order $\omega_{\rm pl}$, and at these frequencies $1/\tau$ and $1/\tau_{\rm eff}$ are very different.

As we pointed out above, in this paper we use another approach and analyze the difference between $1/\tau(\omega)$ in the normal and superconducting states. The leading divergent term in $S(\omega)\sim\omega^2$ at high frequencies is the same for both states. It therefore cancels out in $S_{\rm sc}(\omega)-S_n(\omega)$. If the subleading terms scale as negative powers of frequency at large ω, $S_{\rm sc}(\omega)-S_n(\omega)$ converges and then the sum rule becomes exact. We emphasize that contrary to $S_{\rm eff}(\omega)$, the difference of $S_{\rm sc}(\omega)-S_n(\omega)$, if converges, begins falling off at frequencies that are still much smaller than the fermionic bandwidth and $\omega_{\rm pl}$. In other words, the differential sum rule for the scattering rate is exact in a continuous model.

We now check whether the differential sum rule for $1/\tau$ is satisfied in a BCS superconductor (we argue that it is exact) and also examine the energy scale at which this sum rule is exhausted. In order to explore the issue of convergence we will analyze the differential frequency sums for both $1/\tau(\omega)$ and $\sigma_1(\omega)$ defined as

$$I_\tau(\omega)=\int_0^\omega d\Omega[\tau_{\rm sc}^{-1}(\Omega)-\tau_n^{-1}], \tag{6}$$

$$I_\sigma(\omega)=\int_0^\omega d\Omega[\sigma_{1,{\rm sc}}(\Omega)-\sigma_{1,n}(\Omega)], \tag{7}$$

where $\tau_n=\tau$ and, as we pointed out above, the notations "sc" and "n" refer to superconducting and normal states, respectively.

The expression for the current-current polarization operator in a dirty BCS superconductor has the form

$$\Pi_{\rm sc}(\omega)=\int_0^\infty d\Omega\,\frac{1}{(\sqrt{\Omega_+^2-\Delta^2}+\sqrt{\Omega_-^2-\Delta^2}+i/\tau)}\times\frac{\sqrt{\Omega_+^2-\Delta^2}\sqrt{\Omega_-^2-\Delta^2}-\Delta^2-\Omega_+\Omega_-}{\sqrt{\Omega_+^2-\Delta^2}\sqrt{\Omega_-^2-\Delta^2}}, \tag{8}$$

where $\Omega_\pm=\Omega\pm\omega/2$. In the normal state, this reduces to a conventional Drude form $\Pi_n(\omega)=\omega/(\omega+i/\tau)$. In the superconducting state, one can show quite generally that $\mathrm{Im}\Pi(\omega)$ vanishes below 2Δ. At high frequencies, $\Pi(\omega)$ gradually approaches $\Pi(\infty)=1$.

We first verify whether $S_{\rm sc}(\omega)-S_n(\omega)$ vanishes at $\omega=\infty$. As typical internal frequencies in Eq. (8) are of the same order as external ω, at $\omega\gg\Delta$, one can expand the integrand in Eq. (8) in Δ/ω. The expansion can be straightforwardly carried out for arbitrary $\Delta\tau$, and the result is that at high frequencies

$$\mathrm{Re}\,\Pi_{\rm sc}(\omega)=\mathrm{Re}\,\Pi_n(\omega)\left(1-\frac{1}{1+(\omega\tau)^2}\,\frac{2\Delta^2\ln\dfrac{\omega}{\Delta}}{\omega^2}\right),$$

$$\mathrm{Im}\,\Pi_{\rm sc}(\omega)=\mathrm{Im}\,\Pi_n(\omega)\left(1+\frac{(\omega\tau)^2-1}{(\omega\tau)^2+1}\,\frac{2\Delta^2\ln\dfrac{\omega}{\Delta}}{\omega^2}\right), \tag{9}$$

where $\mathrm{Re}\,\Pi_n(\omega)=(\omega\tau)^2/[1+(\omega\tau)^2]$, $\mathrm{Im}\,\Pi_n(\omega)=-\omega\tau/[1+(\omega\tau)^2]$. Substituting these results into $S(\omega)=\omega^2/\Pi(\omega)$ we find that for large $\omega\gg1/\tau$

$$\mathrm{Re}[S_{\rm sc}(\omega)-S_n(\omega)]\propto\omega^{-2}\quad\mathrm{Im}[S_{\rm sc}(\omega)-S_n(\omega)]\propto\omega^{-1}. \tag{10}$$

We see that both real and imaginary parts of $S_{\rm sc}(\omega)-S_n(\omega)$ vanish at infinite frequency. This implies that Kramers-Kronig transformation is applicable, and hence the differential sum rule is exact for a dirty BCS superconductor.

We now address the issue of the energy scale over which the sum rule is exhausted. We consider clean and dirty limits separately. In the clean limit, the frequency integral in Eq. (8) was evaluated in Ref. 5. To first order in $1/\Delta\tau$ we have for $\omega>2\Delta$

$$\mathrm{Re}\,\Pi_{\rm sc}(\omega)\approx1,\quad \mathrm{Im}\,\Pi_{\rm sc}(\omega)\approx-\frac{1}{\tau\omega}E\left(\sqrt{1-\frac{4\Delta^2}{\omega^2}}\right), \tag{11}$$

where $E(x)=\int_0^{\pi/2}d\phi\sqrt{1-x^2\sin^2\phi}$ is the complete Elliptic integral of the second kind (note that definitions of E differ in different handbooks). In the two limits $E(0)=\pi/2$ and $E(1)=1$. The result for $\mathrm{Re}\,\Pi_{\rm sc}(\omega)\approx1$ is valid outside a tiny $O(1/\Delta\tau)$ range near 2Δ where $\mathrm{Re}\,\Pi_{\rm sc}(\omega)$ diverges logarithmically. Substituting $\Pi_{\rm sc}(\omega)$ from Eq. (11) into $1/\tau_{\rm sc}(\omega)$ we obtain for $\omega>2\Delta$

$$\frac{1}{\tau_{\rm sc}(\omega)}=\frac{1}{\tau}E\left(\sqrt{1-\frac{4\Delta^2}{\omega^2}}\right),\quad \sigma_1(\omega)=\frac{\omega_{\rm pl}^2}{4\pi}\,\frac{1}{\omega^2}\,\frac{1}{\tau_{\rm sc}(\omega)} \tag{12}$$

and $1/\tau(\omega)=\sigma(\omega)=0$ for $\omega<2\Delta$. We plot these functions in Fig. 1. For the differential sum rule we then obtain for $\omega<2\Delta$

$$I_\tau(\omega)=-\frac{\omega}{\tau},\quad I_\sigma(\omega)=\frac{\omega_{\rm pl}^2 1}{4\pi 2\Delta\tau}\left[2\Delta\tau\cot^{-1}\omega\tau-\frac{\pi^2}{8}\right] \tag{13}$$

and for $\omega>2\Delta$

$$I_\tau(\omega)=\frac{2\Delta}{\tau}\int_0^{\omega/2\Delta}dx\left[-1+\mathrm{Re}\,E\left(\sqrt{1-\frac{1}{x^2}}\right)\right], \tag{14}$$

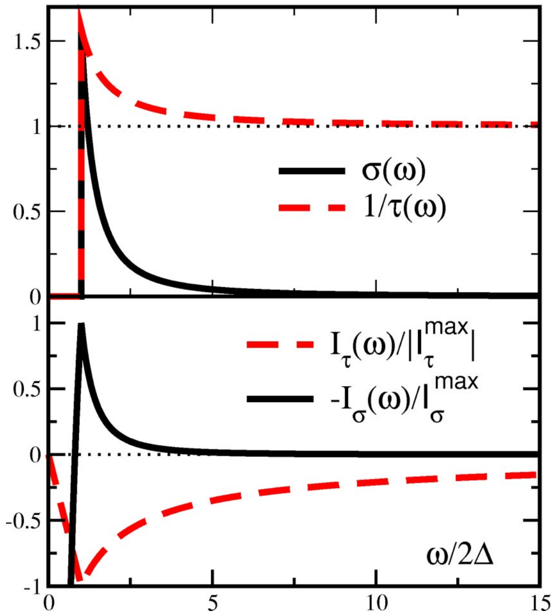

FIG. 1. (Color online) (a) The frequency dependence of the relaxation rate $\tau/\tau_{sc}(\omega)$ and the conductivity $\sigma(\omega)$ in the clean limit. (b) The behavior of $I_\tau(\omega)$ and $I_\sigma(\omega)$ normalized to their maximum values.

$$I_\sigma(\omega)=\frac{\omega_{pl}^2 1}{4\pi 2\Delta\tau}\left[\frac{2\Delta}{\omega}-\frac{\pi^2}{8}+\int_{2\Delta/\omega}^{1}dxE(\sqrt{1-x^2})\right]. \tag{15}$$

We explicitly verified that $I_\tau(\infty)=0$, i.e., the differential sum rule is indeed satisfied.

We also see from Eq. (14) that at finite ω, $I_\tau(\omega)$ depends only on ω/Δ. This implies that in the clean limit, the differential sum rule is exhausted at frequencies $O(\Delta)$. An important issue, however, is how rapidly $I(\omega)$ converges to zero at $\omega\gg\Delta$. In Fig. 1(b) we plot $I_\tau(\omega)$ and $I_\sigma(\omega)$ evaluated numerically from Eq. (14) and normalized to their values at the maximum at $\omega=2\Delta$. We see that $I_\tau(\omega)$ converges much more slowly than $I_\sigma(\omega)$. In particular, at $\omega=15\Delta$, $|I_\tau(\omega)|$ is still about 25% of its maximum value in accord with Ref. 6. On the contrary, $I_s(\omega)$ is vanishingly small at $\omega=15\Delta$. This result fully agrees with Ref. 6. For the same ω, $I_\sigma(\omega)$ practically vanishes.

A weaker convergence of $I_\tau(\omega)$ can be understood analytically. Indeed, at high frequencies, the elliptic function can be expanded in $1/\omega^2$. This yields $1/\tau_{sc}(\omega)=(1/\tau)(1+2(\Delta/\omega)^2[\ln(2\omega/\Delta)-0.5]$. Integrating this expression over frequency, we obtain that at $\omega\gg\Delta$

$$I_\tau(\omega)=-\frac{2\Delta}{\tau}\frac{\Delta}{\omega}[1/2+\ln(2\omega/\Delta)]. \tag{16}$$

The conductivity integral meanwhile converges to zero as

$$I_\sigma(\omega)=-\frac{\omega_{pl}^2}{4\pi}\frac{1}{12\Delta\tau}\left(\frac{2\Delta}{\omega}\right)^3[1/6+\ln(2\omega/\Delta)]. \tag{17}$$

Comparing Eqs. (16) and (17), we see that $I_\sigma(\omega)$ has an extra $(2\Delta/\omega)^2$ that accounts for much faster convergence of $I_\sigma(\omega)$ than of $I_\tau(\omega)$.

We next proceed to the dirty limit $\Delta\tau\ll 1$. As we said in the introduction, the key issue here is whether the sum rule is still exhausted at frequencies $O(\Delta)$, or one needs to extend the integration to frequencies of order $1/\tau$, where the *f*-sum rule for the conductivity is exhausted. At the first glance, in the dirty limit, the frequency integration has to be extended to larger frequencies than in the clean limit, as one can easily show that at $\Delta\tau\ll 1$, the jump of $1/\tau_{sc}(\omega)$ at 2Δ is small, of order $\Delta\tau$, and therefore $1/\tau_{sc}(\omega)$ in a superconductor does not overshoot the normal state $1/\tau$ immediately above 2Δ. However, as we now demonstrate, typical frequencies for the differential sum rule still scale with Δ.

The first indication that the physics is still confined to frequencies $O(\Delta)$ comes from the analysis of the form of $1/\tau_{sc}(\omega)$ at $\omega\sim\tau^{-1}\gg\Delta$. Using Eqs. (9) for $\Pi(\omega)$ and substituting them into $1/\tau_{sc}(\omega)$, we obtain

$$\frac{1}{\tau_{sc}(\omega)}=\frac{1}{\tau}\left(1+\frac{2\Delta^2}{\omega^2}\ln\frac{2\omega}{\Delta}\frac{1+(\omega\tau)^4}{(1+\omega\tau)^4}\right). \tag{18}$$

We see that at frequencies comparable to $1/\tau$, $1/\tau_{sc}(\omega)$ exceeds $1/\tau$, i.e., the overshoot occurs at a lower frequency. Further, integrating $1/\tau_{sc}(\omega)$ from $O(1/\tau)$ to infinity we find that the contribution to $I_\tau(\omega)$ from these frequencies is of order $\Delta^2|\ln(\Delta\tau)|$. Meanwhile, the loss of I_τ in the superconducting state between 0 and 2Δ is $2\Delta/\tau$ that in the dirty limit is much larger than $\Delta^2|\ln(\Delta\tau)|$. This implies that even in the dirty limit the dominant contribution to the sum rule comes from frequencies well below $1/\tau$. At these frequencies, the current-current polarization operator can be evaluated exactly to leading order in $\Delta\tau$ as one can pull $1/\tau$ out of denominator of the integral in the right-hand side of Eq. (8). The remaining integral is evaluated easily, and from substituting the result into $1/\tau_{sc}(\omega)$ we obtain for $\omega>2\Delta$

$$\frac{1}{\tau_{sc}(\omega)}=\frac{1}{\tau}\frac{\omega}{\sqrt{\omega^2-4\Delta^2}}\mathrm{Re}\left[E^{-1}\left(\frac{\omega}{\sqrt{\omega^2-4\Delta^2}}\right)\right],$$

$$\sigma_1(\omega)=\frac{\omega_{pl}^2}{4\pi}\tau\sqrt{1-\frac{4\Delta^2}{\omega^2}}\mathrm{Re}\left[E\left(\frac{\omega}{\sqrt{\omega^2-4\Delta^2}}\right)\right], \tag{19}$$

and $1/\tau_{sc}(\omega)=\sigma_1(\omega)=0$ for $\omega<2\Delta$. We plot these functions in Fig. 2. The behavior of $1/\tau_{sc}(\omega)$ near 2Δ and at high frequencies can be well understood analytically. Near 2Δ, expanding $E(x)$ for large value of the argument, we immediately obtain that $1/\tau_{sc}(\omega)$ evolves continuously [up to corrections $O(\Delta\tau)$], and very near 2Δ behaves as $1/\tau_{sc}(\omega)=(1/\tau)[\omega^2-4\Delta^2]/\omega^2$. Still, it overshoots $1/\tau$ at $\omega\sim 2.68\Delta$ and develops a maximum at $\omega=3.48\Delta$. At larger frequencies, $1/\tau_{sc}(\omega)$ approaches $1/\tau$ as $1/\tau_{sc}(\omega)=(1/\tau)[1+2(\Delta/\omega)^2\ln(2\omega/\Delta)]$, i.e., exactly the same way as in the clean limit. Analyzing the frequency integrals in $I_\tau(\omega)$ and $I_\sigma(\omega)$ we immediately make sure that they converge at ω comparable to 2Δ.

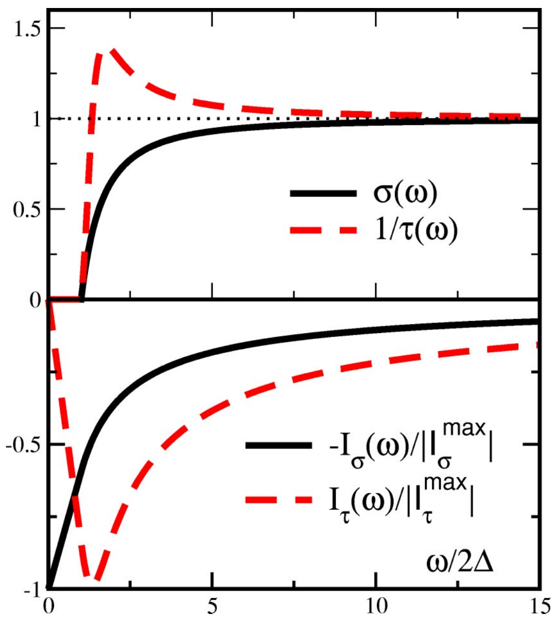

FIG. 2. (Color online) Same as in Fig. 1 but in the dirty limit $\Delta\tau \ll 1$. The conductivity $\sigma(\omega)$ decreases at frequencies comparable to $1/\tau$ (not shown).

For the differential expressions $I_\tau(\omega)$ and $I_\sigma(\omega)$, we obtain after simple manipulations

$$I_\tau(\omega) = -\frac{\omega}{\tau}, \quad I_\sigma(\omega) = \frac{\omega_{\rm pl}^2}{8}\pi\Delta\tau\left[1 - \frac{4\omega}{\pi^2 2\Delta}\right] \tag{20}$$

for $\omega < 2\Delta$, and

$$I_\tau(\omega) = \frac{2\Delta}{\tau}\int_0^{\omega/2\Delta} dx\left[-1 + \frac{x}{\sqrt{x^2-1}}\,\mathrm{Re}\,E^{-1}\left(\frac{x}{\sqrt{x^2-1}}\right)\right],$$

$$I_\sigma(\omega) = \frac{\omega_{\rm pl}^2}{8}\pi\Delta\tau\left[1 - \frac{4}{\pi^2}\frac{\omega}{2\Delta} + \frac{4}{\pi^2}\int_1^{\omega/2\Delta} dx\sqrt{1-\frac{1}{x^2}}\,\mathrm{Re}\,E\left(\frac{x}{\sqrt{x^2-1}}\right)\right] \tag{21}$$

for $2\Delta < \omega \ll 1/\tau$.

In Fig. 2(b) we plot these functions and compare the rates of convergence of $I_\tau(\omega)$ and $I_\sigma(\omega)$ in the dirty limit. We see that, as in the clean limit, $I_\sigma(\omega)$ converges better. We analyzed the high frequency parts analytically and found that a better convergence of $I_\sigma(\omega)$ in the dirty limit is due to the presence of the extra logarithmical term in the high frequency expansion of $I_\tau(\omega)$. We also see that the rate of convergence is almost the same in both clean and dirty limits. We recall that this result is not obvious as in the dirty limit, $1/\tau_{sc}(\omega)$ and $\sigma_1(\omega)$ gradually increases above at 2Δ, while in the clean limit, they both jump at 2Δ and immediately overshoot the normal state values of $1/\tau$ and σ_1, respectively.

Comparing further the behavior of $I_\tau(\omega)$ and $I_\sigma(\omega)$ in clean and dirty limits [Figs. 1(b) and 2(b)], we observe that $I_\tau(\omega)$ does not change much between the two limits, while $I_\sigma(\omega)$ has a different sign at intermediate frequencies in the two limits. Indeed, in the clean limit $I_\sigma(\omega)$ changes sign and becomes negative at $\omega = 16\Delta/\pi^2 < 2\Delta$, while in the dirty limit $I_\sigma(\omega)$ remains positive at intermediate frequencies $2\Delta < \omega \ll 1/\tau$. In particular, we found that at $\omega = 2\Delta$, $I_\sigma(\omega)$ changes sign at $\Delta\tau \approx 1.52$ (it becomes positive at smaller $\Delta\tau$). Still, one can easily make sure that at the highest $\omega \gg 1/\tau$, $I_\sigma(\omega)$ is negative for *arbitrary* $\Delta\tau$, i.e., in the dirty limit, $I_\sigma(\omega)$ should change sign and become negative above some $\omega \sim \gamma$ (not shown in Fig. 2). To verify where this happens we computed analytically $I_\sigma(\omega)$ at $\omega \sim 1/\tau$. The evaluation of $I_\sigma(\omega)$ in this range is tedious but straightforward. We obtained with logarithmical accuracy

$$I_\sigma(\omega) = \frac{\omega_{\rm pl}^2}{2\pi^2}(\Delta\tau)^2|\ln\Delta\tau|Z(\omega\tau), \tag{22}$$

where

$$Z(x) = [\cot^{-1}(x) + 3\tan^{-1}(1/x)] - \frac{2}{x}\frac{2x^2+1}{x^2+1}. \tag{23}$$

Evaluating $Z(x)$ numerically we find that it does indeed change sign at $\omega\tau \sim 0.66$, and is negative for larger ω. This implies that at the highest frequencies, $I_\sigma(\omega)$ approaches zero from below.

Finally, is also instructive to verify explicitly that the differential sum rule for $1/\tau$ is satisfied. Integrating in Eq. (21) over all frequencies by parts using $dE(k)/dk = [E(k) - K(k)]/k^2$ (Ref. 7), where $K(k)$ is the elliptic integral of the first kind, we can rewrite $I_\tau(\infty)$ as

$$I_\tau(\infty) = -\frac{2\Delta}{\tau}\,\mathrm{Im}\int_0^\infty \frac{zK(z)}{E^2(z)}dz. \tag{24}$$

We could not evaluate this integral analytically, but numerical integration yields $I_\tau(\infty) = 0$ with a very high accuracy.

To conclude, in this paper we analyzed the differential sum rule for the scattering rate and optical conductivity in a dirty BCS superconductor. We demonstrated that this sum rule is exact if the normal state τ is independent on temperature. For arbitrary $\Delta\tau$, the sum rule is exhausted at frequencies controlled by Δ, but the convergence is rather weak due to logarithmical terms. We showed that in the dirty limit the convergence of the differential sum rule for the scattering rate is much faster then the convergence of the *f*-sum rule for the conductivity, but slower than for the differential sum rule for conductivity. The latter has the fastest convergence in both clean and dirty limits.

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