A Theory of Auctions with Endogenous Valuations

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April 14, 2018

Brief Summary I

1. Revenue maximization in multi-unit auctions where bidders have non-linear preferences in the allocation
2. Micro-Foundation: auction preceded by costly investments that affect values.
3. In turn, investments affected by the auction $\Longrightarrow$ endogenous values
4. Technical difficulty: "Realization-by-realization" maximization not possible $\Longrightarrow$
5. Relatively complex optimization under feasibility constraint (*reduced auctions*).

Brief Summary II

1. Solution via insights from Majorization Theory, originally due to Hardy, Littlewood and Polya (1929).
2. Main Tool: "*An Integral Inequality*" by Ky Fan and G.G.Lorentz (1954).
3. Main insights:
   - i. sufficient conditions for the optimality of standard auctions;
   - ii. comparative statics about the dependence of the optimal reserve price on demand and supply;
   - iii. how to iron if you must.

The Reduced Form Model

- $m \geq 1$ identical units, and $n \geq m$ ex-ante symmetric bidders.
- Bidder $i \in \{1, \ldots, n\} = N$ has type $\theta_i \in \Theta = [\underline{\theta}, \overline{\theta}]$, demands at most 1 object.
- Types are $I.I.D.$ according to $F : \Theta \to [0, 1]$, density $f > 0$.
- $p_i = p_i(\theta_i)$ is the expected **interim probability** that agent i with type θ_i receives an object in a mechanism.
- The utility of i with type θ_i is given by

$$h(p_i, \theta_i) - y_i$$

  where h is increasing in both variables, super-modular and **convex** in p_i, and where y_i is a monetary payment.
- Assume $h(0, \cdot) = 0$

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Mechanisms

- A *mechanism* specifies a set of reports R_i for each agent, and a mapping from reports to an allocation and transfers:

$$x : \prod_{i \in N} R_i \to X = [0,1]^n \; ; \;\; y : \prod_{i \in N} R_i \to \mathbb{R}^n \, .$$

  Given a mechanism (x, y) agent i picks an optimal report r_i.
- We restrict attention to *symmetric mechanisms* that are invariant to permutations of agents' names.

Micro-Foundation: Costly Investments

- Agent i takes private action $a_i \in A \subseteq \mathbb{R}$, where A is compact set.
- Depending on θ_i, i has preferences over:
  1. her action $a_i \in A$;
  2. her allocation x_i, $x \in X = \{0,1\}^n$;
  3. her transfer y_i.
- Utility function of the form:

$$x_i\, v(a_i, \theta_i) - y_i - c(a_i)\,.$$

  where $c(a)$ is the cost of a.

- c are increasing in a ; v is is super-modular, non-negative and increasing in a and θ.
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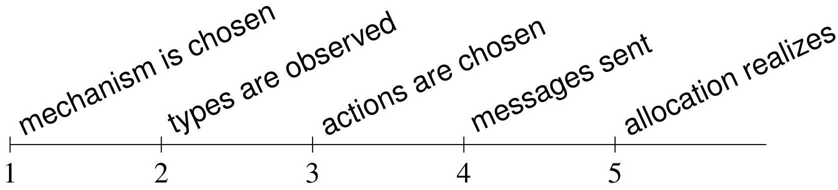
Micro-Foundation: Timing



1. The designer commits to a mechanism;
2. Each agent privately observes her type;
3. Each agent privately chooses an action;
4. Each agent sends a message to the mechanism;
5. Depending on the messages, an allocation and transfers are realized.

Micro-Foundation: Reduction to Non-linear Preferences

- Studied in Kreps and Porteus (1979) and Machina (1984).
- Given mechanism (x, y), agent i conditions her action on the report she plans to send. Reporting problem equivalent to

$$\max_{r_i \in R_i} \left(\left\{ \max_{a_i \in A} \mathbb{E}\left[x_i(r) \mid \theta_i\right] v(a_i, \theta_i) - c(a_i) \right\} - \mathbb{E}\left[y_i(r) \mid \theta_i\right] \right) .$$

.

- Define

$$p_i(r_i) = \mathbb{E}\left[x_i(r_i, r_{-i})\right] .$$

to be the interim probability with which i receives an object, and

$$h(p_i, \theta_i) = \max_{a_i \in A} p_i \, v(a_i, \theta_i) - c(a_i) .$$

to be the utility i receives when she takes the optimal action.
- The reporting problem of agent i is equivalent to the "reduced form" problem where i has non-linear preferences over p_i .

Incentive Compatibility & Revenue

- **Lemma:** *It follows from*

$$\frac{\partial^2 h(p,\theta)}{\partial p \partial \theta} \geq 0$$

that incentive compatibility $\Leftrightarrow$ *monotonicity of* p *together with an envelope condition on interim expected utility.*

- **Lemma:** *The revenue in any symmetric, IC mechanism where the participation constraint is binding for the lowest type is given by*

$$n \int_{\Theta} H(\theta, p(\theta)) f(\theta)\, d\theta\,.$$

where the generalized "virtual value" $H : [\underline{\theta}, \overline{\theta}] \times [0,1] \to \mathbb{R}$ *is defined by*

$$H(\theta, p) := h(p(\theta), \theta) - h_\theta(p(\theta), \theta) \times \frac{1 - F(\theta)}{f(\theta)}\,.$$

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The Feasibility Constraint

- **Theorem** *(Che,Kim,Mierendorff, 2013): A symmetric, interim allocation rule* $q : [0,1] \to [0,1]$ *represents a feasible allocation iff for each* $t \in [0,1]$,

$$\int_t^1 q(s)ds \leq \frac{1}{n} \sum_{i=0}^{n} \min\{i,m\} \binom{n}{i} (1-t)^i \, t^{n-i} \, .$$

- **Lemma:** *It holds that*

$$\sum_{i=0}^{n} \min\{i,m\} \binom{n}{i} t^{n-i} (1-t)^i = n \int_t^1 \phi_{m,n}(t)dt$$

where

$$\phi_{m,n}(t) = \sum_{i=0}^{m-1} \binom{n-1}{i} t^{n-1-i} (1-t)^i$$

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The Revenue Maximization Problem

- Define quantile $t = F^{-1}(\theta)$, probability $q(t) = p(F^{-1}(t))$, virtual value $G(t, q) := H(F^{-1}(t), q)$.
- The revenue maximization problem (for symmetric mechanisms) is given by:

$$\max_q \int_0^1 G(t, q(t))dt \text{ subject to:}$$

$$q(t) \in [0, 1] \text{ for all } t \in [0, 1] \quad (1)$$

$$q \text{ non-decreasing} \quad (2)$$

$$\int_t^1 q(s)ds \leq \int_t^1 \phi_{m,n}(t)dt, \text{ for all } t \in [0, 1] \quad (3)$$

Majorization

- For non-decreasing $p, q \in L^1(0,1)$ we say that q **majorizes** p, denoted by $p \prec q$ if:

$$1.\ \int_t^1 p(v)dv \leq \int_t^1 q(v)dv \text{ for all } t$$

$$2.\ \int_0^1 p(t)dt = \int_0^1 q(t)dt$$

- We say that q **weakly majorizes** p, denoted by $p \prec_w q$ if 1. holds (but not necessarily 2.).
- If $p \prec_w q$ there exists $q' \leq q$ such that $p \prec q'$.
- $p \prec q$ if and only if $p = Tq$ where T is a doubly stochastic operator.

The Fan-Lorentz Integral Inequality I

- **Theorem:** *Let* $L(t,p)$ *be a real-valued function defined on* $[0,1]\times[0,1]$ *such that*
    1. L is convex in p;
    2. L is super-modular in (t,p);

    Let $p,q:\ [0,1]\to[0,1]$ *be two non-decreasing functions such that* $p\prec q$. *Then*

$$\int_0^1 L(t,p(t))dt \le \int_0^1 L(t,q(t))dt$$

The Fan-Lorentz Integral Inequality II

- The *orbit* $\Omega(q) = \{p : p \prec q\}$ is weakly compact and convex.
- By the Bauer Maximum Principle (1958) a continuous, convex functional on $\Omega(q)$ attains its maximum on an extreme point of $\Omega(q)$.
- $p \in \Omega(q)$ is extreme iff $p = q \circ \omega$ where ω is a measure-preserving transformation of $[0, 1]$ (Ryff, 1967).
- FL assert that all convex functionals satisfying $\frac{\partial^2 L}{\partial t \partial p} \geq 0$ attain their maximum on $\Omega(q)$ at

$$q \circ Id = q.$$

The Optimal Mechanism

- **Theorem:** *Suppose that the "virtual utility" H satisfies*

$$\frac{\partial^2 H}{(\partial p)^2} \geq 0 \text{ and } \frac{\partial^2 H}{\partial\theta\partial p} \geq 0$$

Then the optimal allocation awards the m objects to the agents with the highest types, conditional on these exceeding a threshold $\theta^\star$ that satisfies

$$H(\psi_{m,n}(\theta^*), \theta^*) = 0.$$

where $\psi_{m,n} := \phi_{m,n} \circ F$.

Proof (Sketch, uniform distribution) :

- For every feasible p , $p \prec_w \phi_{m,n}$.
- This implies that $p \prec 1_{[\widehat{\theta},1]} \times \phi_{m,n}$ for some $\widehat{\theta} \in [0,1]$.
- Hence, by the FL Theorem

$$\int_0^1 H(p(\theta),\theta)d\theta \leq \int_0^1 H((1_{[\widehat{\theta},1]} \times \phi)(\theta),\theta)d\theta$$

- Revenue maximizer has the form $1_{[\hat{\theta},1]} \times \phi_{m,n}$.
- As $H(0,\theta) = 0$ the problem becomes

$$\max_{\hat{\theta}} \int_{\hat{\theta}}^1 H(\phi_{m,n}(\theta),\theta) \,.$$

- Thus, the first order condition is

$$H(\phi_{m,n}(\theta^\star),\theta^\star) = 0 \,.$$

- Verifying that the objective is quasi-concave yields the result. □

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The Linear Special Case

- Recall the "regular", linear case of Myerson (1979) where

$$
\begin{aligned}
H(p,\theta) &= p\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) \\
\frac{\partial^2 H}{(\partial p)^2} &= 0;\ \frac{\partial^2 H}{\partial\theta\partial p} = \frac{d}{d\theta}\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) \geq 0
\end{aligned}
$$

The optimal cut-off type (equals the reserve price in standard auctions) is independent of the number of agents and objects in the linear framework!

Comparative Statics : Optimal Cut-Off

Theorem: *The optimal cut-off type defined by*

$$H(\psi_{m,n}(\theta^\star), \theta^\star) = 0.$$

increases in the number of agents n *and decreases in the number of objects* m.

- **Proof:** Follows from the properties of $\phi_{m,n}(\theta)$ and the implicit function theorem.
- Intuition (Micro-Foundation): An increase in n decreases the chance of getting an object. In turn, this reduces investments and values, Hence the revenue from each individual.
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Theorem: *The optimal cut-off type defined by*

$$H(\psi_{m,n}(\theta^\star), \theta^\star) = 0.$$

increases in the number of agents n *and decreases in the number of objects* m.

- **Proof:** Follows from the properties of $\phi_{m,n}(\theta)$ and the implicit function theorem.
- Intuition (Micro-Foundation): An increase in n decreases the chance of getting an object. In turn, this reduces investments and values, Hence the revenue from each individual.
- In particular, the type that yields zero revenue must go up.
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Implementation via Uniform-Price Auctions

Theorem: *Assume that the seller uses a uniform $(m+1)-$price auction with reserve price $\mathcal{R}$, and define $\theta'_{m,n}$ by:*

$$\frac{h\left(\psi_{m,n}(\theta'_{m,n}), \theta'_{m,n}\right)}{\psi_{m,n}(\theta')} = \mathcal{R}$$

1. *The profile of bidding strategies*

$$b_i(\theta) = b_{m,n}(\theta) = \begin{cases} \frac{\partial h(\psi_{m,n}(\theta),\theta)}{\partial p} & \theta \geq \theta'_{m,n} \\ 0 & \theta < \theta'_{m,n}\,. \end{cases}$$

   constitutes a symmetric, pure strategy equilibrium.
2. *For each fixed type θ, the equilibrium bid $b_{m,n}(\theta)$ increases in m and decreases in n.*
3. *The uniform price auction is revenue maximizing if the reserve price is set to*

$$\mathcal{R}^* = \frac{h\left(\psi_{m,n}(\theta^*), \theta^*\right)}{\psi_{m,n}(\theta^*)}$$

Implementation via Pay-Your-Bid Auctions

Theorem: *Assume that the seller uses a discriminatory pay-your-bid price auction with reserve price $\mathcal{R}$, and define $\theta'_{m,n}$* by*:*

$$\frac{h\left(\psi_{m,n}(\theta'_{m,n}), \theta'_{m,n}\right)}{\psi_{m,n}(\theta')} = \mathcal{R}$$

1. The profile of bidding strategies

$$\beta(\theta) = \begin{cases} \mathcal{R}\frac{\psi_{m,n}(\theta')}{\psi_{m,n}(\theta)} + \frac{1}{\psi_{m,n}(\theta')}\int_{\theta'}^{\theta_i} \psi'_{m,n}(z)\frac{\partial h(\psi_{m,n}(z),z)}{\partial p}dz & \theta \geq \theta' \\ 0 & \theta < \theta' \end{cases}.$$

constitutes a symmetric, pure strategy equilibrium.
2. The discriminatory auction is revenue maximizing if the reserve price is set to

$$\mathcal{R}^* = \frac{h\left(\psi_{m,n}(\theta^*), \theta^*\right)}{\psi_{m,n}(\theta^*)}$$

Comparative Statics: Optimal Reserve Price

Theorem: *Assume that the environment is convex super-modular, that* $\frac{\partial^2 h}{\partial \theta^2} \leq 0$*, and that* F *is convex and twice differentiable. Then the optimal reserve price*

$$\mathcal{R}^* = \frac{h\left(\psi_{m,n}(\theta^*), \theta^*\right)}{\psi_{m,n}(\theta^*)}$$

decreases in the number of agents n *and increases in the number of objects* m*.*

Illustration: Additively Separable Investments

- Assume that v satisfies:

$$v(a, \theta) = a + \theta$$

- Then

$$h(p, \theta) = \max_{a \in A} p\,(a + \theta) - c(a)$$

.

- Take an arbitrary selection $a^\star(p) \in \arg\max_{a \in A} p\,a - c(a)$. Then, h is given by

$$h(p, \theta) = p\theta + pa^\star(p) - c(a^\star(p)) = p\theta + g(p)\,,$$

where

$$g(p) = pa^\star(p) - c(a^\star(p))$$

is convex.

Revenue with Additively Separable Investments

- Expected Revenue:

$$n \int \left[p(\theta)(\theta - \frac{1 - F(\theta)}{f(\theta)}) + g(p(\theta)) \right] f(\theta) d\theta \, .$$

- Note that

$$\begin{aligned} \frac{\partial^2 H}{(\partial p)^2} &= g''(p) \geq 0 \text{ and} \\ \frac{\partial^2 H}{\partial \theta \partial p} &= \frac{d(\theta - \frac{1-F(\theta)}{f(\theta)})}{d\theta} \geq 0 \end{aligned}$$

where the latter holds if the standard virtual value is increasing.

Optimal Reserve Price with Additively Separable Investments

Corollary :

1. *Assume that* $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ *is increasing.Then both the Uniform-Price and the Pay-Your-Bid auction with reserve price*

$$\mathcal{R}^{*} = \frac{1 - F(\theta^{*})}{f(\theta^{*})}$$

where $\theta^{\star}$ *solves*

$$\theta^{*} + \frac{g(\psi_{m,n}(\theta^{\star}))}{\psi_{m,n}(\theta^{\star})} = \frac{1 - F(\theta^{*})}{f(\theta^{*})}$$

are symmetric, revenue maximizing mechanisms.

2. *If, in addition, the hazard rate* $\frac{f(\theta)}{1-F(\theta)}$ *is increasing, then the optimal reserve price decreases in the number of agents* n *and increases in the number of units* m.

Other Environments

- Similar analysis for multiplicative values

$$a \in \mathbb{R}_+,\ v(a,\theta) = a \cdot \theta,\ c(a) = b \cdot a^l / l$$

- or the case with entry cost

$$a \in \{0,1\},\ v(a,\theta) = a \cdot \theta,\ c(a) = b\,a\,.$$

Relaxing Super-modularity

- If $\frac{\partial H}{\partial\theta\partial p} \geq 0$ does not hold, we may get other extreme points as maximizers of the FL inequality.

- Need to get insight into the extreme points of the set

$$\Omega_{\mathbf{mon}}(\phi) = \left\{ p : p \prec \phi = \sum_{i=0}^{m-1} \binom{n-1}{i} t^{n-1-i}(1-t)^i \ \wedge \ p \text{ monotone} \right\}$$

- **Lemma:** *If p is an extreme point of $\Omega_{\mathbf{mon}}(\phi)$ and $p' \neq 0$ on an interval, then $p = \phi$ on that interval.*

Ironing

- Assume p extreme point of $\Omega_{\mathbf{mon}}(\phi)$, $p = d$ on $[\theta_1, \theta_2]$ and $p = \phi$ elsewhere.
- By majorization, $p(\theta_2) < \phi(\theta_2)$. If $p < \phi$ on $[\theta_1, \theta_2]$, then p cannot be an extreme point.
- Assume then that $p(\theta_3) = q(\theta_3)$ for $\theta_3 \in (\theta_1, \theta_2)$. Then, for p to be extreme it must hold that:

$$\begin{aligned}
\int_{\theta_1}^{\theta_3} (d - \phi(\theta)) d\theta &= \int_{\theta_3}^{\theta_2} (\phi(\theta) - d) d\theta \Leftrightarrow \\
d &= \frac{\int_{\theta_1}^{\theta_2} \phi(\theta) d\theta}{\theta_2 - \theta_1}
\end{aligned}$$

Conclusion

- Optimal auction design in an environment with non-linear preferences
- Combined mechanism design techniques with tools from majorization theory.
- Obtained new insights about incentives in auctions preceded by costly actions that affect values.
- Illuminated and generalized old results.
- Asymmetric mechanisms ? Information Acquisition ?