Job Market Paper

Dynamic Monitoring Design

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This paper introduces flexible endogenous monitoring in dynamic moral hazard. A principal can commit to not only an employment plan but also the monitoring technology to incentivize dynamic effort from an agent. Optimal monitoring follows a Poisson process that produces rare informative signals, and the optimal employment plan features increasing entrenchment. To incentivize persistent effort, the Poisson monitoring takes the form of "bad news" that leads to immediate termination. Monitoring is non-stationary: the bad news becomes more precise and less frequent. When persistent effort is not required, the optimal incentive scheme features a trial period of non-stationary monitoring, and a combination of Poisson bad news that leads to termination and Poisson good news that leads to tenure.

Keywords: dynamic monitoring, moral hazard, information design
JEL classification: D82, D83, D86

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1 Introduction

Moral hazard is central to economics. The past two decades have seen the proliferation of research that significantly enriches our understanding in dynamic moral hazard problems in employment relationships, organizational structures, and regulatory policies. With few exceptions, existing work focuses on incentive provision under exogenous monitoring technologies. In many applications, however, the monitoring technology is a critical endogenous component of incentive design. The goal of this paper is to study dynamic incentive provision *without* restrictions on the monitoring technology. I study a dynamic moral hazard model in which the principal monitors the agent's effort by acquiring informative signals, and adapts the incentive scheme contingent on the acquired signals. The monitoring design raises two connected questions: "which signals to acquire" and "how to adapt to the acquired signals". For example, a medical board can incentivize a physician to provide quality service by acquiring either a strong signal of malpractice that leads to delicensing, or weak signals that in multitude lead to suspension; a supervisor can incentivize a worker to exert effort by adapting future employment to past evaluations.

In the model, the principal (she) has full commitment to monitoring the agent's (his) binary private effort subject to a monitoring cost, and adapting future employment contingent on past signals. The wage is fixed. I model flexible public monitoring by a Blackwell experiment on the agent's effort. I assume the monitoring cost function satisfies likelihood ratio separability (Assumption 1) and compound reduction (Assumption 2), with the relative entropy function as the leading example.

I consider first the main model in which the principal must incentivize effort when the agent is employed. I show that the optimal monitoring takes the form of Poisson bad news that leads to immediate termination, i.e., the agent is never employed again (Theorem 1). The optimal incentive scheme features minimal history dependence: conditional on the agent being employed, the non-stationary monitoring depends only on the length of employment. Over time, the Poisson bad news becomes more precise and less frequent. The agent's continuation value increases with the length of employment because termination becomes a more effective incentive instrument. The frequency decreases so quickly that the relationship continues indefinitely with positive probability. My results are consistent with Mincer and Jovanovic (1981) who find that the length of employment explains half of the variation in termination, and the hazard rate of termination (turnover) is decreasing.

The non-stationarity follows from the main tradeoff between the principal's costly monitoring and information exposure to the agent. To provide incentives, the principal can acquire more precise signals or adapt future employment more sensitively to acquired signals. On one hand, more precise signals are more costly to the principal. On the other hand, more sensitive adaption means that the agent's continuation value is more volatile in the sense of second-order stochastic dominance. The public signal exposes more information about his continuation value to the agent, and allows him to devise more elaborate deviations from the recommended effort. The information exposure imposes a cost to the principal because she faces more stringent incentive constraints to prevent such deviations. As the agent's continuation value increases, the public signal exposes more information about the continuation value to the agent. To reduce this cost, the principal decreases the public signal's information about the continuation value by substituting the frequency by the precision of arrival.

My analysis makes use of two salient features of flexible endogenous monitoring in stark contrast with exogenous monitoring. The first feature is that the principal can probabilistically mix informative monitoring with uninformative signals, on which she chooses not to adapt the incentive scheme, i.e., the scheme continues as if the uninformative signal has not been acquired. The uninformative signal exposes no payoff-relevant information to the agent and thus creates no additional incentive constraints. I show that, as the agent takes action more often, the principal's value weakly increases and the mixed monitoring converges to Poisson monitoring (Lemma 1). This contrasts with exogenous monitoring models in which players must adapt to frequent signals of limited informativeness. The players' values decrease because they observe more signals *informative* about the continuation play and create more incentive constraints, as shown by Abreu, Milgrom, and Pearce (1991) in partnership games.

The second feature is that the principal can acquire more precise signals by aggregating multiple signals *across periods* into a one-off signal. Because more precise signals warrants greater change in the incentive scheme, the principal combines multiple signals to make sure that some signal is precise enough to warrant immediate reaction in the form of termination (Lemma 2). It is also optimal to pool all other signals that do not warrant immediate reaction in order to expose less payoff-relevant information to the agent. This explains the minimal history dependence in the optimal incentive scheme. The endogenous signal aggregation contrasts exogenous monitoring models in which the principal must accumulate signals of limited precision *over time* for signals precise enough to warrant reaction. Before termination, the incentive scheme depends sensitively on the history of accumulated signals.

I consider then an extension in which the principal needs not incentivize effort when employing the agent. The extension allows shirking as a new incentive instrument. I show that the optimal incentive scheme takes one of four possible forms (Theorem 2). They feature Poisson bad news that leads to termination and Poisson good news that leads to tenure, e.g., permanent shirking. The monitoring shows increasing precision and decreasing frequency during a trial period of deterministic length. The first two forms are up-or-out schemes, i.e., the agent becomes either terminated or tenured by the end of the trial period. The first form uses Poisson bad news in the trial period during which the agent's continuation value increases so much that, absent arrivals, he obtains tenure at the end. The second form uses Poisson good news in the trial period during which the agent's continuation value decreases so much that, absent arrivals, he gets terminated at the end. As tenure becomes a more effective incentive instrument, it also exposes more payoff-relevant information to the agent and so the principal substitutes the frequency of good news with precision. The last two form features stationary two-sided Poisson monitoring (bad news that leads to termination and good news that leads to tenure) after the trial period. The third form uses Poisson bad news during the trial period, and the fourth form uses Poisson good news.

Related literature

My paper incorporates flexible endogenous monitoring to the dynamic moral hazard problem with imperfect monitoring. Pioneering papers including Rubinstein (1979) and Rogerson (1985) formulate the problem by repeated games with stationary exogenous monitoring. DeMarzo and Sannikov (2006) and Sannikov (2008) formulate the incentive provision problem in continuous time and introduce the martingale representation approach. One key insight from this literature is that the continuation of incentive scheme depends not only on the history of actions but also the history of signals because players need to accumulate signals of limited precision over time. With endogenous monitoring, however, I show that the optimal incentive scheme features minimal history dependence in that, contingent on the agent being employed, the continuation of the incentive scheme only depends on the length of employment.

The role of monitoring in dynamic moral hazard problems is discussed as early as in Abreu, Milgrom, and Pearce (1991). They show in a partnership game that, when players observe exogenous signals more often, the set of equilibrium values shrinks due to the cost of information exposure. Sannikov and Skrzypacz (2010) show that Brownian monitoring and Poisson monitoring provide incentives in different ways in a continuous-time partnership game. Fudenberg and Levine (2007, 2009) and Sadzik and Stacchetti (2015) study how the details of the monitoring technology in discrete time affect the incentive provision and value of incentive schemes at the continuous-time limit. I contribute to this literature by resolving the tradeoff between costly monitoring and information exposure.

Recent developments in dynamic incentive provision incorporate restricted forms of endogenous monitoring.[1] Marinovic, Skrzypacz, and Varas (2018) and Varas, Marinovic, and Skrzypacz (2020) endogenize the timing of monitoring with costly state verification; Piskorski and Westerfield (2016) the frequency of conclusive Poisson bad news; Fahim, Gervais, and Krishna (2021) the precision of Brownian monitoring. In particular, Dai, Wang, and Yang (2021) endogenize the direction of conclusive Poisson news of fixed frequency and find that, when the agent has low continuation value, the optimal incentive scheme monitors with *good news* because the conclusive bad news is not frequent enough to incentivize effort. In contrast, by endogenizing the frequency and precision as well, I find that the optimal incentive scheme monitors with frequent but inconclusive *bad news*. This highlights how restrictions in endogenous monitoring can affect qualitative predictions.

[1] Georgiadis and Szentes (2020) and Li and Yang (2020) study optimal static incentive schemes with flexible endogenous monitoring.

The flexible monitoring technology in my model relates to the literature of information design. Kamenica and Gentzkow (2011) introduces the Bayesian persuasion problem and belief-based approach. Ely (2017), Ely and Szydlowski (2020), Hébert and Zhong (2022), and Koh and Sanguanmoo (2022) use the posterior belief to study dynamic persuasion problems. See Bergemann and Morris (2019) for a survey. The belief-based approach is also used to study rational inattention problems under posterior separable attention costs (Caplin and Dean, 2015). Morris and Strack (2019) study the relation between attention cost and sequential sampling. Ravid (2020) studies a game with rationally inattentive players and finds that unreasonable equilibria arise from the possibly degenerate belief over endogenous actions. To overcome this problem, I model the monitoring by the distribution of likelihood ratios and the monitoring cost by an "experimental cost" (Denti, Marinacci, and Rustichini, 2022). I adapt properties of attention costs to the monitoring cost: posterior separability to likelihood ratio separability (Assumption 1) and sequential learning proofness (Bloedel and Zhong, 2020) to compound reduction (Assumption 2). Using the belief-based approach, Zhong (2022) and Georgiadis-Harris (2021) study dynamic information acquisition under a posterior separable cost.

Notably, Zhong (2022) is the most closely related paper in information design. He studies the Wald problem of information acquisition before a one-off decision, and shows the optimality of Poisson signals that lead to immediate decision. His decision problem differs from my incentive provision problem which features the agent's incentive compatibility constraint. In his model, Poisson signal induces the riskiest decision time and thus maximizes expected utility under risk-loving exponential discounting. The immediate decision follows because, with higher continuation value, the impatient decision maker faces higher opportunity cost of not taking the one-off decision. His intuition, however, does not apply to my incentive provision problem due to flow payoffs and the additional dimension of future employment.

My model provides a moral hazard theory for the negative empirical relationship between termination and length of employment, complementing the existing search theory (Burdett, 1978; Jovanovic, 1984) and experience theory (Jovanovic, 1979).[2]

[2] Jovanovic (2021) finds a decreasing hazard rate of product recall in a moral hazard model in which reputation has direct utility consequence.

See Gibbons and Waldman (1999) for a survey.

2 Dynamic monitoring model

I model the dynamic incentive provision problem in continuous time. The principal commits to a signal history-dependent incentive scheme which consists of the monitoring technology to acquire public signals about the agent's private effort, and the contingent plan to adapt future employment decisions to the signal history. The agent has no commitment and chooses whether to exert effort when he is employed.

I introduce the timeline of the dynamic incentive scheme and then formulate the principal's design problem. Heuristically, the stage game at each instant $t \in [0, \infty)$ in continuous time follows the timeline below.

1. The principal publicly chooses whether to employ the agent for the instant $h_t \in \{0, 1\}$. The stage game ends if she chooses not to employ $h_t = 0$.
2. The principal publicly chooses a costly monitoring.
3. The agent privately chooses whether to exert costly effort $a_t \in \{0, 1\}$.
4. The chosen monitoring generates a public signal about the current private effort.

2.1 Monitoring technology and monitoring cost

The principal chooses the monitoring technology that specifies how to monitor the agent's private effort based on past signals. I model the monitoring technology by a càdlàg martingale $\mathbf{\Gamma}$ with $\mathbf{\Gamma}_0 = 0$ that specifies the cumulative likelihood ratio $\mathbf{\Gamma}_t - \mathbf{\Gamma}_s$ during time interval $(s, t]$. For such monitoring technology, I define the cumulative monitoring cost up to time t as a stochastic process

$$C_t(\mathbf{\Gamma}) := \limsup_{\Delta t \to 0} \sum_{m=1}^{\lceil t/\Delta t \rceil} C\left(1 + \mathbf{\Gamma}_{m\Delta t} - \mathbf{\Gamma}_{(m-1)\Delta t}\right)$$

where $\lceil \cdot \rceil$ rounds up to the nearest integer and C is the monitoring cost function. In this section, I shall explain how continuous-time processes $\mathbf{\Gamma}$ and $C_t(\mathbf{\Gamma})$ model the monitoring technology and monitoring cost by introducing their discrete-time counterparts.

In discrete time, the principal monitors the agent's current effort by choosing a Blackwell experiment, which can be represented by a distribution of likelihood ratio subject to the Bayes rule. A Blackwell experiment specifies the distribution of signal $\mathbb{P}^a$ for each of binary private effort $a \in \{0,1\}$ in the current period. A signal is informative about the private effort only through its likelihood ratio[3] $L := d\mathbb{P}^{a=0}/d\mathbb{P}^{a=1} \in (0,\infty)$. Moreover, a distribution of likelihood ratio corresponds to a Blackwell experiment if and only if it satisfies the Bayes rule $\mathbb{P}^{a=1}[L] = 1$. Therefore, I represent the Blackwell experiment by likelihood ratio distribution $\mathbf{L} \in \Delta_1(0,\infty)$ and its signal by likelihood ratio $L \sim \mathbf{L}$, where $\Delta_1(0,\infty)$ denotes the set of probability measures on $(0,\infty)$ with expectation one.

The Blackwell experiment incurs a monitoring cost to the principal. The monitoring cost function C maps each experiment $\mathbf{L} \in \Delta_1(0,\infty)$ to its cost $C(\mathbf{L}) \in [0,\infty]$. I make two two main assumptions on the non-parametric monitoring cost: *likelihood ratio separability* and *compound reduction*. My leading example is the relative entropy cost function $C(\mathbf{L}) = \mathbb{E}_{L\sim\mathbf{L}}[-\log(L) + L - 1]$.

Likelihood-ratio separability states that the monitoring cost is linear in probability—it is a convex moment of the distribution of likelihood ratio.

Assumption 1 (Likelihood-ratio separability) *There exists convex $\mathcal{C}^2$ function $c : (0,\infty) \to [0,\infty)$ such that $C(\boldsymbol{L}) = \mathbb{E}_{L\sim\mathbf{L}}[c(L)]$ for all $\mathbf{L} \in \Delta_1(0,\infty)$.*

Cost function c is convex so that the monitoring cost is monotonic in the Blackwell order, i.e., a more precise monitoring is more costly. I normalize c such that $c(1) = c'(1) = 0$. One interpretation of separability is that each signal L costs $c(L)$, so that the monitoring cost of an experiment equals the expected cost of the realized signal when the agent exerts effort.

Compound reduction states that compound monitoring is no cheaper than the

[3] I assume away perfectly informative signals $L = 0, \infty$, which will be infinitely costly.

reduced monitoring.

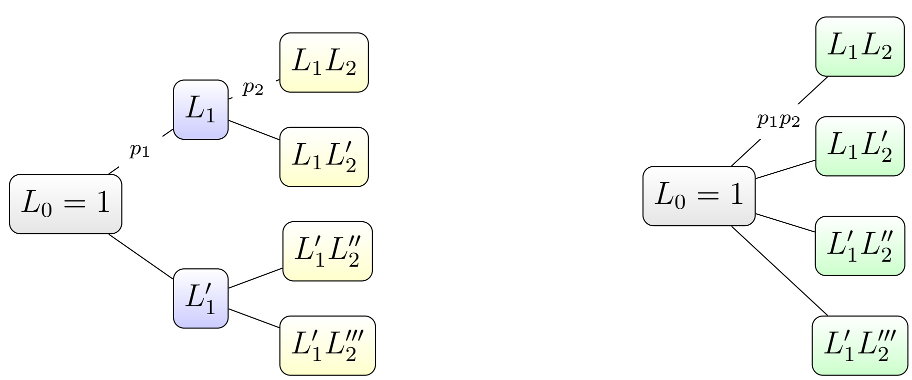
Assumption 2 (Compound reduction) *For all* $\boldsymbol{L}_1 \in \Delta_1(0,\infty)$ *with finite support, and* $\boldsymbol{L}_2 : \operatorname{supp}(\boldsymbol{L}_1) \to \Delta_1(0,\infty)$*, the monitoring costs satisfy*

$$C\left(\mathbf{L}_1\right) + \mathbb{E}_{L_1 \sim \mathbf{L}_1}\left[C\left(\mathbf{L}_2(L_1)\right)\right] \geq C\left(\mathbf{L}_1 \times \mathbf{L}_2(\mathbf{L}_1)\right)$$

where $\mathbf{L}_2(\mathbf{L}_1)$ *is the mixture distribution of* $\mathbf{L}_1$ *and* $\mathbf{L}_2$*.*

The assumption concerns a hypothetical scenario where the principal monitors the same private effort twice (Figure 1). Conditional on first signal L_1, the principal independently monitors the effort again to obtain the second signal L_2, so that the likelihood ratio of the two monitoring together is the product of the two likelihood ratios $L_1 L_2$. Compound reduction states that the compound monitoring, which generates L_1 and then L_2, costs no less than the reduced monitoring, which generates the product likelihood ratio $L_1 L_2$ directly. One interpretation is that the monitoring cost is the reduced form of the least costly compound monitoring for a given distribution of product likelihood ratios. Note that compound monitoring costs the same as reduced monitoring under the relative entropy cost function; see Pomatto, Strack, and Tamuz (2018).

In addition to likelihood ratio separability and compound reduction, I assume the Inada condition $\lim_{L\to 0,\infty} c'(L)(L-1) - c(L) = \infty$ to guarantee the existence of an



(a) Compound monitoring. (b) Reduced monitoring.

Figure 1: Compound monitoring and the corresponding reduced monitoring.

optimal incentive scheme. See Mirrlees (1999) for an example of non-existence.

I represent the discrete-time monitoring technology by the cumulative likelihood ratio and define the cumulative monitoring cost by partial sums. Let $\mathbf{L}_m$ denote the monitoring and L_m the realized signal in period m. The monitoring may depend on past signals. I write the cumulative likelihood ratio up to period n as a stochastic process $\mathbf{\Gamma}_n := \sum_{m=1}^{n}(\mathbf{L}_m - 1)$. It is a discrete-time martingale because the likelihood ratio has expectation one. The cumulative monitoring cost up to period n is a stochastic process that can be written in terms of the martingale difference $\{\mathbf{\Gamma}_m - \mathbf{\Gamma}_{m-1} : m \leq n\}$

$$\sum_{m=1}^{n} C(\mathbf{L}_m) = \sum_{m=1}^{n} C\left(1 + \mathbf{\Gamma}_m - \mathbf{\Gamma}_{m-1}\right) =: C_n(\mathbf{\Gamma})$$

where $\mathbf{\Gamma}_0 := 0$. I write $C(\mathbf{L}) = \infty$ if $\mathbf{L} \notin \Delta_1(0, \infty)$. For any discrete-time martingale $\mathbf{\Gamma}$ such that $C_n(\mathbf{\Gamma})$ is almost-surely finite for all n, there exists a sequence of monitoring $\{\mathbf{L}_m\}$ adapted to past signals such that $\mathbf{\Gamma}_n := \sum_{m=1}^{n}(\mathbf{L}_m - 1)$.

Any (continuous-time) monitoring technology $\mathbf{\Gamma}$, such that $C_t(\mathbf{\Gamma})$ is finite almost surely, is the limit of a sequence of the discrete-time counterparts $\{\mathbf{L}_{\Delta t,m} : m \leq \lceil t/\Delta t \rceil\}$ in that $\sum_{m=1}^{\lceil s/\Delta t \rceil}(\mathbf{L}_{\Delta t,m} - 1) \to \mathbf{\Gamma}_s$ in distribution for all $s \leq t$.

2.2 Payoffs from employment, effort, and monitoring

The principal and agent derive flow payoffs from employment, effort, and monitoring. Moral hazard arises as the principal earns revenue from the agent's costly private effort.

The agent derives utility from being employed and incurs an effort cost. When employed $h_t = 1$, he derives a constant flow utility $u > 0$ which can be interpreted as the benefit of being employed or a fixed wage. He also incurs an effort cost $k > 0$ in exerting effort $a_t = 1$. When he is not employed $h_t = 0$, the agent's flow payoff is normalized to zero. The agent's von Neumann-Morgenstern payoff is then

$$\int_0^{\infty} re^{-rt} h_t(u - ka_t)dt$$

where $r > 0$ is the discount rate common to both the principal and agent. I assume the agent prefers being employed and exerting effort over not being employed, i.e., $u - k > 0$.

The principal derives utility from the agent's private effort and incurs the monitoring cost. When employing the agent $h_t = 1$, she earns flow revenue $\pi > 0$ if the agent exerts effort $a_t = 1$, and zero if not $a_t = 0$. The principal does not observe her revenue; she can only infer from her monitoring. She incurs incremental monitoring cost $dC_t(\Gamma)$ for monitoring Γ over small time interval Δt. When not employing the agent, the principal's flow payoff is zero.[4] The principal's von Neumann-Morgenstern payoff is then

$$\int_0^\infty e^{-rt} h_t \left(r\pi a_t dt - dC_t(\Gamma) \right) .$$

2.3 Dynamic monitoring problem

The principal commits to a dynamic incentive scheme to maximize her expected payoff subject to the agent's incentive compatibility.

A dynamic incentive scheme $\mathcal{M}$ consists of

- filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ which satisfies the usual conditions and to which calendar time t is adapted,
- monitoring technology $\mathbf{\Gamma}$ which is a càdlàg martingale with $\mathbf{\Gamma}_0 = 0$,
- predictable employment decision h and effort recommendation a.

The filtered probability space can be larger than the natural filtration of the monitoring technology and calendar time to accommodate public randomizations. The probability law $\mathbb{P}$ corresponds to the case in which the agent always exerts effort. Because the signal realization depends on the private effort, I denote by $\mathbb{P}^{a'}$ the law

[4]In other words, the principal does not pay to employ the agent. Costly employment does not qualitatively change the results.

under predictable effort[5] a'.

The principal's problem is to choose a dynamic incentive scheme $\mathcal{M}$ to maximize her expected payoff

$$\mathbb{E}^a \left[\int_0^\infty e^{-rt} h_t \left(r\pi a_t dt - dC_t(\Gamma) \right) \right]$$

subject to the agent's incentive compatibility constraint

$$a \in \max_{a'} \mathbb{E}^{a'} \left[\int_0^\infty re^{-rt} h_t (u - k a'_t) dt \right] .$$

I denote the principal's value of incentive scheme $\mathcal{M}$ by $V(\mathcal{M})$. I restrict the principal's choice to continuous-time incentive schemes that can be approximated by discrete-time incentive schemes in value. Formally, there exists a sequence of discrete-time incentive schemes $\{\mathcal{M}_{\Delta t}\}$ that converges to the continuous-time incentive scheme in value $\lim_{\Delta t \to 0} V_{\Delta t}(\mathcal{M}_{\Delta t}) = V(\mathcal{M})$, where $V_{\Delta t}(\mathcal{M}_{\Delta t})$ is the principal's value in $\mathcal{M}_{\Delta t}$. See Appendix A.1 for the discrete-time monitoring problem.

Remark 1 My restriction connects the tractable continuous-time abstraction to the well-grounded economic models in discrete time. As Fudenberg and Levine (2007, 2009) and Sadzik and Stacchetti (2015) point out under exogenous monitoring with Brownian motion, continuous-time incentive schemes need not be a good approximation of discrete-time incentive schemes with short time periods. My restriction rules out pathological cases such as the "infinite switches" equilibrium of Keller, Rady, and Cripps (2005).[6] A sufficient condition for the restriction is that the monitoring technology $\mathbf{\Gamma}$ is a simple or compound Poisson process of bounded frequency.

[5]I define $\mathbb{P}^{a'}$ as the extension to the change of measure $\frac{d\mathbb{P}^{a'}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t^{a'}$ where $Z_t^{a'}$ is the stochastic exponential of the martingale $\int (1 - a'_t) d\Gamma_t$, i.e., $dZ_t^{a'} = Z_{t-}^{a'}(1 - a'_t) d\Gamma_t$ and $Z_0^{a'} = 1$. The extension exists and is unique by the Girsanov theorem.

[6]See the discrete time version of that model in Hörner, Klein, and Rady (2022).

3 Optimal dynamic incentive scheme

To highlight the economic forces, I first study the main model in which the principal must incentivize effort when the agent is employed, e.g., $h_t = 1 \implies a_t = 1$. I defer the extension with the possibility of shirking to Section 4.

3.1 Main result

The optimal dynamic incentive scheme uses Poisson monitoring—the cumulative likelihood ratio $\mathbf{\Gamma}$ is a compensated Poisson process parameterized by Poisson jump size $\Delta\Gamma_t \in (-1, \infty)$ and bounded frequency λ_t, and the filtered probability space is the augmented natural filtration of $\mathbf{\Gamma}$ and time t. The likelihood ratio upon Poisson arrival is given by $L_t := \Delta\Gamma_t + 1 \in (0, \infty)$ due to normalization. I call such monitoring technology Poisson bad news if $L_t > 1$ because the arrival is more likely when the agent does not exert effort, and I call it Poisson good news if $L_t < 1$. With abuse of notation, I refer to the arrival by Poisson bad news/good news. I say a Poisson news is more precise if L_t is further away from 1.

I state the optimal dynamic incentive scheme and then elaborate on its properties.

Theorem 1 *In the optimal dynamic incentive scheme,*

- *the principal monitors the agent's effort by Poisson bad news that leads to immediate termination;*
- *conditional on no bad news arrival, the Poisson monitoring increases in precision, decreases in frequency, and eventually increases in monitoring cost. The frequency decreases so quickly that the agent is employed indefinitely with positive probability.*

The optimal incentive scheme features minimal history dependence. Because the agent is terminated upon one bad news arrival, the signal history must be an interval of "no arrivals" conditional on the agent being employed. Calendar time corresponds one-to-one with this unique history, and therefore the likelihood ratio and frequency of the optimal Poisson monitoring, conditional on no arrival, are functions of the length of employment.

The optimal dynamic incentive scheme is non-stationary in that the likelihood ratio and frequency of Poisson bad news depends on the length of employment (Figure 2). It is instructive to contrast it with a stationary incentive scheme in which the principal monitors with stationary Poisson bad news that leads to termination.

The agent's continuation value is increasing because a more punishing termination eases incentive provision. For illustration, suppose the optimal incentive scheme is stationary so that the agent's continuation value is constant. Flexible monitoring allows us to consider a one-step deviation: a short and small increase in frequency and decrease in precision such that incentive compatibility is preserved. To compensate for the additional risk of termination, the agent's continuation value increases conditional on no arrival. After the one-step deviation, the principal resumes the stationary monitoring with decreased frequency for this increased continuation value.

The deviation yields the principal higher value and thus contradicts the optimality

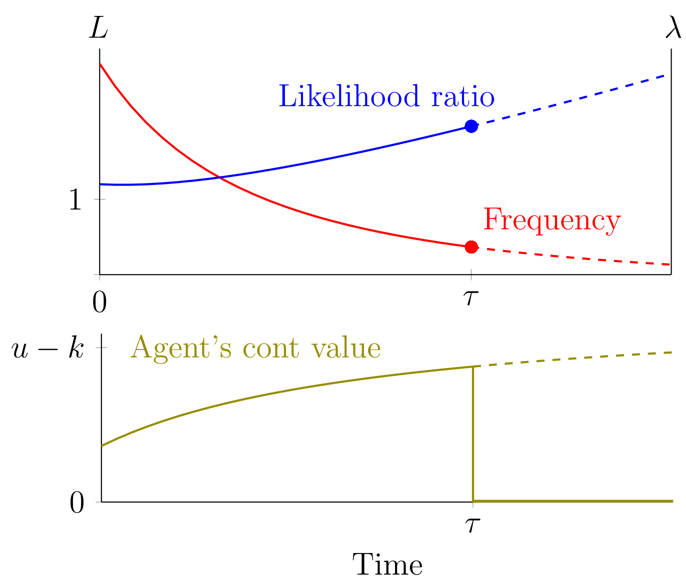


Figure 2: A realization of the optimal dynamic incentive scheme. The Poisson bad news arrives at τ.

of stationary incentive schemes. By supposition, the stationary monitoring is optimal so that the small changes in monitoring entail a second-order increase in monitoring cost during the deviation. The additional probability of termination during the deviation is exactly compensated by the decrease in the future, and therefore the principal receives the same revenue from the agent's effort. However, when the agent has more to lose from termination, the principal can incentivize effort with less frequent bad news which decreases future monitoring costs. The decrease is of the first-order because termination gives the minimum continuation value.

The increasing precision, decreasing frequency, and eventually increasing monitoring cost follow from the tradeoff between costly monitoring and information exposure. For fixed frequency λ, when the agent's continuation value is higher, the public signal is more informative about the continuation value in the sense of second-order stochastic dominance—with arrival at frequency λ, the continuation value experiences a bigger drop to zero upon termination; with no arrival, the continuation value experiences a bigger upward drift proportional to λ to compensate for the risk of termination. Exposed to such improved information, the agent can devise more elaborate deviations which are more costly for the principal to prevent. To reduce the cost of information exposure, the principal is willing to expend greater monitoring costs to substitute the frequency by the precision of Poisson monitoring.

Because of compound reduction, the Poisson monitoring decreases in frequency so quickly that, with positive probability, the agent is employed and exerts effort indefinitely. Assumption 2 implies that the marginal cost c' increases with likelihood ratio L so slowly that it is bounded from above $\lim_{L\to\infty} c'(L) < \infty$. As the agent accumulates continuation value, the principal finds it less expensive to monitor with explosively precise bad news with vanishing frequency. This contrasts with exogenous monitoring models in which the likelihood ratio is bounded from 0 and ∞ such that the agent is terminated eventually.[7]

Following Spear and Srivastava (1987), the optimal dynamic incentive scheme

[7] Due to bounded likelihood ratio, the sensitivity of the agent's continuation value to the likelihood ratio is bounded from below. No-shirking implies the existence of uniform finite time and positive probability such that the agent is terminated within the stated time with at least that probability, *regardless of his continuation value*. The Borel–Cantelli lemma thus implies the eventual termination.

admits a recursive formulation with the agent's continuation value W as the state variable. The continuation value W_t is the payoff the agent expects to derive after time t

$$W_t := \mathbb{E}_t \left[\int_t^{\infty} r e^{-r(s-t)} h_s \left(u - k a_s \right) ds \right].$$

With abuse of notation, I denote by $V(W)$ the principal's value function, which is concave and attains value zero at termination $V(0) = 0$. As W ranges from 0 to $u - k$, the likelihood ratio L of Poisson bad news monitoring increases from the uninformative 1 to the conclusive ∞, and frequency λ decreasing from ∞ to 0.

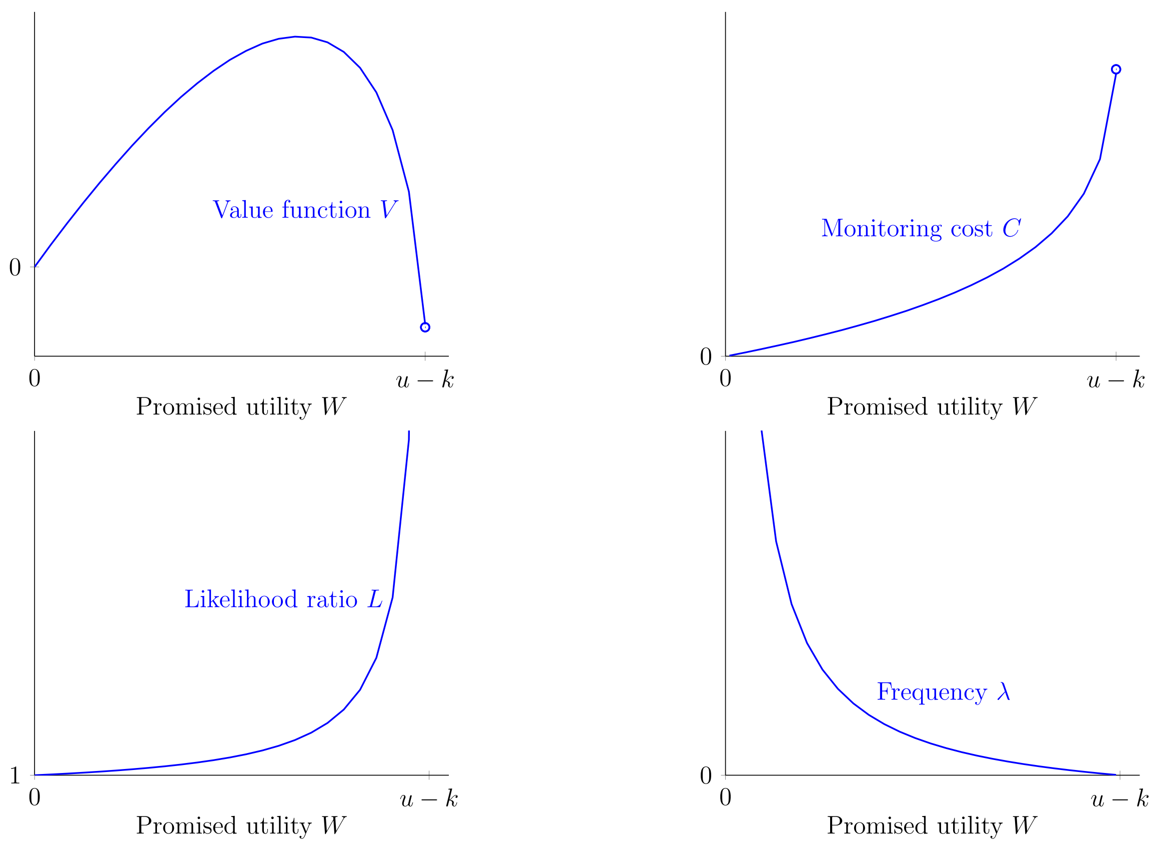


Figure 3: Value function and optimal monitoring.

3.2 Overview of proof strategy

I highlight the main challenges and overview my strategy to derive the optimal incentive scheme.

The dynamic monitoring problem presents two main challenges. First, the rich space of monitoring technologies prevents direct applications of the dynamic programming principle. In particular, I cannot establish an Hamilton-Jacobi-Bellman (HJB) equation by comparing with discrete-time Bellman equations because the agent's flow payoff can contribute to incentive provision.[8] Second, the evolution of dynamic incentive scheme is endogenously determined by the cost of information exposure, which is in turn determined by future evolutions of the scheme.

I derive the optimal dynamic incentive scheme in four steps by leveraging the discrete-time counterpart. First, the optimal discrete-time scheme does not use public randomization, which exposes information to the agent without providing incentives. As a consequence, optimal signals lead to either immediate termination or continued employment and monitoring in the next period. Second, Poisson monitoring is sufficient to maximize the principal's value. For any discrete-time schemes, I construct a continuous-time incentive scheme with compound Poisson monitoring by mixing the uninformative signal with informative monitoring, in a way that does not expose the agent to more information despite more frequent observations. This allows me to establish an HJB equation of Poisson monitoring about the value function. Third, I show that, in the optimal discrete-time incentive scheme, there exists a positive probability signal that leads to immediate termination. Otherwise, the principal can delay costly monitoring and expose less information by pooling the agent's information sets by aggregating monitoring across periods, which decreases the monitoring cost by compound reduction. The immediate termination in discrete time implies that the principal's value function satisfies an HJB equation of immediate termination upon Poisson arrival, and thus resolves the evolution of the optimal incentive scheme. Fourth, I construct a candidate value function, verify its optimality, and derive the optimal incentive scheme by solving a first-order ordinary differential equation about

[8] In contrast, Zhong (2022) studies flexible dynamic information acquisition and manages to establish an HJB equation by comparing with discrete-time Bellman equations because the decision maker faces no incentive constraints and derives no flow payoffs.

the optimal likelihood ratio.

In the remainder of this section, I shall elaborate on the sufficiency of Poisson monitoring and the existence of signal that leads to immediate termination because these two steps highlight the roles of flexible monitoring and information exposure in dynamic incentive provision.

3.3 Sufficiency of Poisson monitoring

Within the rich space of monitoring technologies, Poisson monitoring is sufficient to maximize the principal's value. As a result, the value function satisfies an HJB equation of Poisson monitoring.

Proposition 1 *The value function V is a viscosity solution to HJB equation*

$$rv(W) = \sup_{\lambda,L,J} r\pi + r(W-u+k)v'(W) + \lambda\left(v(J) - v(W) - (J-W)v'(W)\right) - \lambda c(L)$$

subject to instantaneous incentive compatibility constraint

$$\lambda(1-L)(J-W) = rk \tag{1}$$

for $W \in (0, u-k)$ and boundary condition $v(0) = 0$.

See Definition 3 in the appendix for the definition of viscosity solution.

Under Poisson monitoring, the incentive scheme specifies three control variables: frequency λ and likelihood ratio L of Poisson monitoring, and the agent's continuation value upon arrival, jump J, which is the state variable for the incentive scheme. The control needs to satisfy the agent's instantaneous incentive compatibility (IC), which is the continuous-time version of the one-step deviation principle. Intuitively, the continuation value must decrease $J < W$ for Poisson bad news $1 - L < 0$, and vice versa. The frequency λ needs to be sufficiently high to provide enough incentives to overcome flow effort cost rk.

The HJB equation decomposes the principal's value into four terms. The first term is the flow revenue $r\pi$ derived from the agent's effort. The second term is the change in the principal's value due to the expected change in the agent's continuation value. To account for the promised value the principal owes to the agent, W_t grows at interest rate r and falls due to the flow payoff $r(u-k)$. The expected change translates to the principal's value by marginal value $V'(W)$. The third term is the cost of information exposure. Mathematically, the cost equals the expected change in the principal's value from the mean-preserving spread in the agent's Poisson continuation value. With arrival at frequency λ the principal's value jumps from $V(W)$ to $V(J)$, and with no arrival it drifts by $-\lambda(J-W)V'(W)$. The fourth term is the cost of Poisson monitoring $\lambda c(L)$.

The optimal controls (λ, L, J) follow from an incentive-cost analysis. They provide incentives in IC (1) but incurs the monitoring cost and the cost of information exposure in the HJB equation. The incentive and costs are linear in frequency λ due to expected utility and likelihood ratio separability, and so the optimal likelihood ratio and jump maximize the incentive-cost ratio

$$(L^*, J^*) \in \arg\max_{L,J} \frac{(1-L)(J-W)}{-\left(V(J)-V(W)-V'(W)(J-W)-c(L)\right)}. \tag{2}$$

The optimal frequency then follows from binding IC.

For fixed jump J, the optimal likelihood ratio L features the intratemporal tradeoff between the monitoring cost and the cost of information exposure, summarized by a first-order condition (FOC)

$$c'(L)(L-1)-c(L) = -\left(V(J)-V(W)-(J-W)V'(W)\right). \tag{3}$$

When the principal optimally combines the monitoring technology and contingent plan of future employment, the marginal monitoring cost per incentive provided equals the marginal cost of information exposure. When the jump J is further from W, the public signal is more informative about the continuation value in the sense of second-order stochastic dominance. The cost of information exposure increases and so the principal expends greater monitoring costs for more precise signals in order to reduce the frequency of payoff-relevant information exposure.

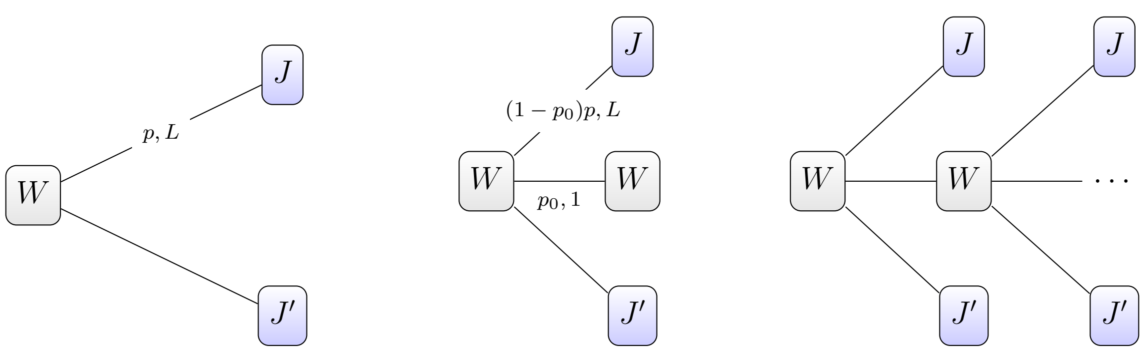
The key idea of Proposition 1 is that continuous-time incentive schemes with compound Poisson monitoring can replicate discrete-time ones in value. The intuition is that mixing uninformative signals with informative monitoring does not expose additional information or create new incentive compatibility constraints, and such mixing converges to compound Poisson monitoring at the continuous-time limit. Analogous to Poisson monitoring, compound Poisson monitoring is a monitoring technology in which $\mathbf{\Gamma}$ is a compound Poisson process of bounded frequency, and the filtered probability space is the augmented natural filtration of $\mathbf{\Gamma}$ and time t.

Lemma 1 (Poisson replication) *For any discrete-time incentive scheme, there exists a continuous-time incentive scheme with compound Poisson monitoring that gives strictly higher value.*

The sufficiency of compound Poisson monitoring implies the sufficiency of Poisson monitoring because a compound Poisson process is a convex combination of compensated Poisson processes. This is the main reason for the continuous-time formulation of the dynamic incentive provision problem.

I shall show that, with shorter time periods, the principal can increase her value by mixing the uninformative signal with informative monitoring. For short period length $\Delta t > 0$, take a Δt-incentive scheme at continuation value W. The principal uses monitoring $\mathbf{L}$ and continuation value $\mathbf{J}$ to satisfy Δt-IC (Figure 4a)

$$e^{-r\Delta t}\mathbb{E}_{\mathbf{L},\mathbf{J}}\left[(1-L)(J-W)\right] = (1-e^{-r\Delta t})k\,.$$



(a) Contract in Δt.

(b) Mixing in uninformative signal in $\Delta t'$.

(c) Iteratively mixing in uninformative signals in Δt.

Figure 4: Replicating a incentive scheme's value in shorter time periods.

Analogous to the continuous-time IC (1), the covariation between the likelihood ratio and continuation value equals to the effort cost for the Δt-period. Now, consider the problem with the period length halved $\Delta t' := \Delta t/2$. In each $\Delta t'$-period, the effort cost and derived revenue are also halved approximately.[9] To half the incentive provision, the principal can mix the uninformative signal ($L \equiv 1, J \equiv W$) with probability $p_0 \approx 1/2$, and conduct informative monitoring $(\mathbf{L}, \mathbf{J})$ only with probability $1 - p_0$ such that the $\Delta t'$-IC binds[10] (Figure 4b)

$$e^{-r\Delta t'} \left(p_0 \times 0 + (1 - p_0) \times \mathbb{E}_{\mathbf{L},\mathbf{J}} \left[(1 - L)(J - W)\right]\right) = (1 - e^{-r\Delta t'})k \,.$$

With the effort cost and probability of reward/punishment $\mathbf{J}$ halved, the mixed incentive scheme offers the same expected value to the agent. Due to common discounting, the principal also gets the same expected value from effort and future continuations. The monitoring cost decreases by more than half because the principal incurs the cost only $\Delta t' < \Delta t$ before conditioning on the signals.

I show the sufficiency of compound Poisson monitoring by constructing a dynamic incentive scheme with compound Poisson monitoring (Figure 4c). At any initial continuation value W, the principal mixes the uninformative signal with informative monitoring until the first informative signal arrives (Figure 4c). At new continuation value J, the principal mixes again until an informative signal arrives. Because discrete time is countable, the iteration constructs a dynamic incentive scheme for the shorter period[11] $\Delta t'$. As $\Delta t' \to 0$, this incentive scheme converges to a continuous-time incentive scheme with compound Poisson monitoring because the probability of an informative signal is proportional to the period length[12] $\Delta t'$.

Endogenous monitoring allows mixing with the uninformative signal which does not expose payoff-relevant information, and thus increases the principal's value with shorter time periods. Despite the more frequent opportunities to deviate, the agent

[9] They are slightly more than half because of the convex exponential discounting.

[10] In fact, $p_0 < 1/2$ because, conditional on an informative signal, continuation value J arises sooner and thus provides a slightly stronger incentive.

[11] I show in the appendix that optimal monitoring in discrete time consists of finitely many signals.

[12] The decrease in monitoring cost vanishes as $\Delta t \to 0$ so that the constructed scheme can be approximated in discrete time.

does not observe additional information. In other words, the possibility of uninformative signal in endogenous monitoring preserves the agent's incentive constraint. In contrast, under exogenous monitoring, Abreu, Milgrom, and Pearce (1991) find that the value of a partnership game decreases with shorter time periods because players observe more *informative* signals which create more incentive constraints.

3.4 Signal leading to immediate termination

I shall show that, in the optimal incentive scheme in discrete time, there exists a positive-probability signal that leads to immediate termination. The intuition is that the incentive scheme must adapt immediately to justify the costly monitoring and information exposure; if not, the monitoring should be delayed.

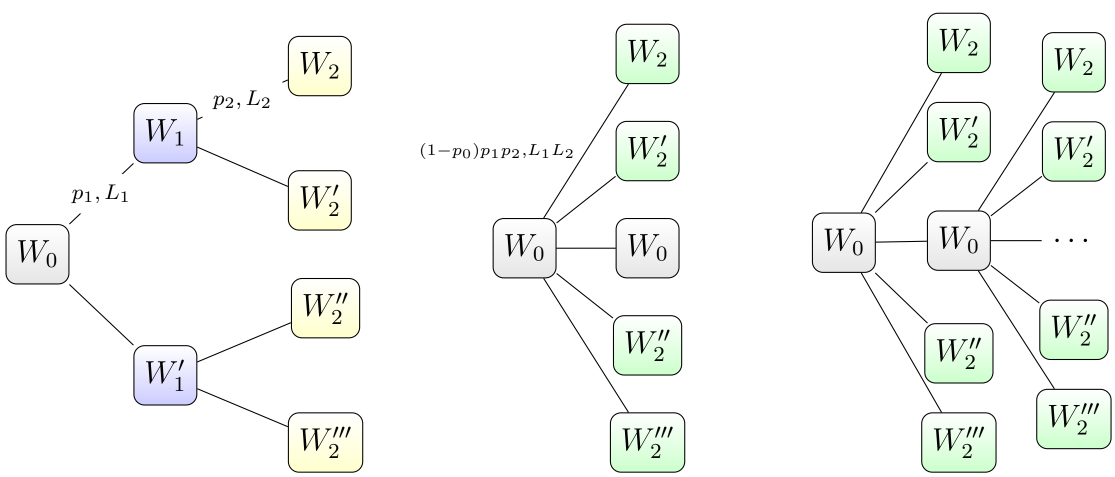
Lemma 2 (Immediate reaction) *In any optimal discrete-time incentive schemes, there is a positive-probability signal that leads to immediate termination.*

Although periods beyond the next can adapt to the current signal to provide incentives, the *optimal* incentive scheme must react to some signal through immediate termination.

To prove Lemma 2 by contradiction, I suppose, at some initial continuation value W_0, every possible first-period signal L_1 leads to the corresponding W_1 at which the principal continues to employ the agent and acquires a second-period signal L_2 that leads to W_2 (Figure 5a). I shall construct an alternative incentive scheme that yields strict higher value by delaying cost monitoring and information exposure. The alternative scheme mixes two kinds of monitoring. With probability p_0, the first is the uninformative signal that leads to the same continuation value W_0. With probability $1 - p_0$, the second is the reduced monitoring that aggregates across the two periods such that signal $L_1 L_2$ leads to W_2 (Figure 5b). The compound reduction is possible because the effort choice is the same across the two periods. Probability p_0 is chosen such that the agent's continuation value remains the promised W_0.

I show that the alternative incentive scheme is incentive compatible and offers strictly higher value to the principal by decreasing the monitoring cost. Pooling the agent's information at initial W_0 and the intermediate W_1's, the alternative scheme inherits incentive compatibility by removing the intermediate information exposure. Due to common discounting, probability p_0 implies that the principal also derives the same revenue on expectation. The monitoring cost decreases for two reasons. First, the reduced monitoring incurs weakly lower monitoring cost due to compound reduction (Assumption 2). Second, the alternative scheme delays the first costly signal L_1 by one period on expectation. As shown in Figure 5c, the scheme acquires the reduced signal $L_1 L_2$ just one period, instead of two, before reacting to it at W_2. The delayed monitoring saves on the temporally discounted cost.

In contrast with exogenous monitoring, the optimal incentive scheme under endogenous monitoring must react to some signal immediately to justify costly monitoring and information exposure. If no signal is precise enough to warrant immediate termination, the principal can increase the precision by reducing the monitoring across time periods (and mixing in the uninformative signal to preserve incentives) until some signal is precise enough. This will delay the costly monitoring and remove information exposure to the agent in the intermediate periods. Under exogenous monitoring, however, the exogenous signals may not be precise enough to warrant reaction so that the principal can only accumulate these signals over time, exposing



(a) Incentive scheme without immediate reaction.

(b) Compound reduction in one period.

(c) Compound reduction in multiple periods.

Figure 5: Immediate reaction and compound reduction in dynamic incentive schemes.

payoff-relevant information to the agent in the intermediate periods.

4 General effort recommendations

In contrast with no-shirking incentive schemes in the main model, I study an extension in which the principal can recommend either exerting effort or shirking. I show that the minimal history dependence is robust, and decreasing turnover generalizes to monotonic hazard rate of termination and tenure.

Under some parameter values, the optimal dynamic incentive scheme makes use of stationary two-sided Poisson monitoring—a compound Poisson monitoring of two possible arrivals, bad news and good news, with stationary frequencies and likelihood ratios.

Theorem 2 *Depending on model parameters, the optimal incentive scheme takes one of four possible forms. All forms feature Poisson monitoring, the possibility of tenure, and a trial period of deterministic duration. Over the trial period, the Poisson monitoring becomes more precise and less frequent. The four forms can be categorized into up-or-out schemes and eventually stationary schemes.*

1. *The first form is an up-or-out scheme in which the principal monitors with Poisson bad news that leads to termination during the trial period and, absent arrivals, the agent obtains tenure at the end.*

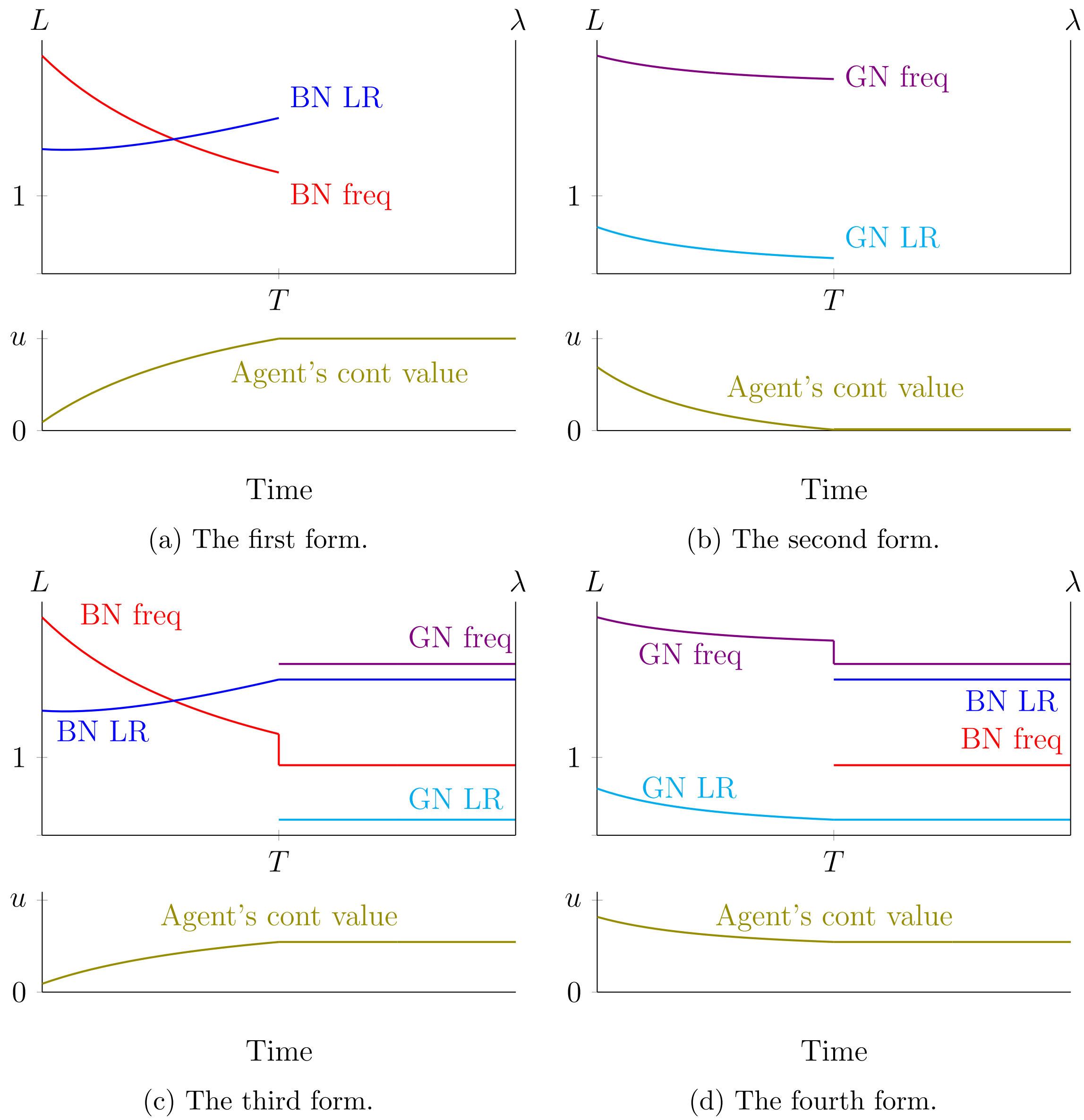
2. *The second form is an up-or-out incentive scheme in which the principal monitors with Poisson good news that leads to tenure during the trial period and, absent arrivals, the agent gets terminated at the end.*

3. *The third form is an eventually stationary incentive scheme in which the principal monitors with Poisson bad news that leads to termination during the trial period and switches to stationary two-sided Poisson monitoring—bad news leads to termination and good news leads to tenure.*

4. *The fourth form is an eventually stationary incentive scheme in which the principal monitors with Poisson good news that leads to tenure during the trial pe-*

riod and switches to stationary two-sided Poisson monitoring—bad news leads to termination and good news leads to tenure.

All possible incentive schemes feature minimal history dependence (Figure 6).



(a) The first form. (b) The second form.

(c) The third form. (d) The fourth form.

Figure 6: The likelihood ratio (LR) and frequency (freq) over time in the four possible forms of incentive schemes. I denote the duration of the trial period by T. The agent's continuation value jumps to 0 upon Poisson bad news (BN) and to u upon Poisson good news (GN).

The agent is tenured or terminated upon Poisson arrival and therefore, condition on the principal monitoring the agent's effort, the signal history must be an interval of "no arrivals". Minimal history dependence is optimal because information exposure beyond the effort recommendation is costly to the principal.

During the trial period, the principal uses one of two types of monitoring, Poisson bad news or Poisson good news, and the monitoring becomes more precise and less frequent over time. In case of Poisson bad news, the agent's continuation value is increasing because a more punishing termination eases incentive provision, as in the main model. The principal substitutes frequency by precision to reduce the cost of information exposure. Symmetrically, in case of Poisson good news, the continuation value is decreasing because a more rewarding tenure eases incentive provision, and the principal substitutes frequency by precision.

In up-or-out schemes, the agent is either tenured or terminated by the end of the trial period. This feature is observed in employment relationships in accounting, consulting, and law firms and in academia. In the first form, the continuation value increases absent bad news arrivals until it attains the maximum u, i.e., the agent obtains tenure, at the end of the trial period (Figure 6a). The hazard rate of termination is decreasing over the trial period. In the second form, the continuation value decreases absent good news arrivals until it attains the minimum 0, i.e., the agent gets terminated, at the end of the trial period (Figure 6b). The hazard rate of tenure is decreasing over the trial period.

In the eventually stationary schemes, the principal switches to stationary two-sided Poisson monitoring at the end of the trial period. In the third form, the continuation value increases absent bad news arrivals, until it reaches a threshold at which Poisson good news monitoring becomes equally optimal (Figure 6c). When the principal switches to stationary two-sided Poisson monitoring, the likelihood ratio of bad news is continuous because of the continuous cost of information exposure, but the frequency of bad news decreases discontinuously because the additional good news monitoring supplements incentive provision. The stationarity can be interpreted as chattering between bad news monitoring and good news monitoring. When the continuation value is below the threshold, the principal monitors with Poisson bad news so that, absent arrivals, the continuation value increases above the threshold. Now that the continuation value is above the threshold, the principal monitors with Poisson good news so that, absent arrivals, the continuation value decreases below the threshold. The chattering continues until one of the Poisson news arrives. The hazard rate of termination is decreasing, continuously over the trial period and discontinuously at the end, and the hazard rate of tenure is increasing discontinuously at the end. The fourth form is analogous to the third form with Poisson good news instead of bad news during the trial period so that the continuation value decreases (Figure 6d). The hazard rate of tenure is decreasing and the hazard rate of termination is increasing.

The intuition for the optimal incentive schemes in the extension is analogous to the main model. The sufficiency of Poisson monitoring follows from the same replication of discrete-time schemes by continuous-time schemes with compound Poisson monitoring. The optimality of termination or tenure upon Poisson arrival follows because the principal must react to some signal by either terminating or tenuring the agent immediately. I narrow down the optimal dynamic scheme to the four possible forms using the evolution of continuation value absent arrivals. If Poisson bad news monitoring is initially optimal, the continuation value increases condition on no arrival. It reaches either the maximum u at which the agent obtains tenure (the first form), or a threshold at which Poisson good news monitoring and thus stationary two-sided monitoring are equally optimal (the third form).[13] Symmetrically, the second and fourth form follow if Poisson good news monitoring is initially optimal.

5 Conclusion

This paper provides a dynamic incentive provision framework which allows flexible monitoring design, and characterizes the optimal incentive scheme. My first contribution is an optimization foundation for a simple class of monitoring technologies of rare informative signals. My second contribution is to derive the optimal dynamic incentive scheme using the optimality of immediate reaction. The optimal scheme features minimal history dependence due to the cost of information exposure. The

[13] I show that the rate of increase is bounded from below.

non-stationary scheme provides a moral hazard theory of the decreasing turnover observed in employment relationships.

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A Proof of Theorem 1

The proof strategy for Theorem 1 follows four steps. First, I derive properties of optimal discrete-time incentive schemes. Second, I establish an HJB equation by replicating discrete-time incentive schemes with compound Poisson incentive schemes. Third, I show the optimality of immediate reaction which implies that the continuation value upon Poisson arrival is termination. Fourth, I derive the optimal incentive scheme by explicitly constructing the value function and then verifying its optimality.

A.1 Discrete-time incentive provision

In this subsection, I define the discrete-time incentive provision problem and then study the properties of the optimal incentive scheme. The key results are that a canonical incentive scheme, that does not randomize and monitors with at most four signals until the agent is terminated, is optimal (Lemma 13) and that the discrete-time value functions converge at the continuous-time limit (Corollary 1).

A.1.1 Discrete-time problem

I define the discrete-time incentive provision problem because the principal's chosen continuous-time incentive scheme needs to be approximated by discrete-time incentive schemes in value.

For time unit $\Delta t > 0$, the stage game of the dynamic incentive scheme follows the timeline in Section 2 in each period $n \in \{1, 2, ...\}$.

A discrete-time incentive scheme consists of a complete filtered probability space $(\Omega, \{\mathcal{F}_n\}_n, \mathbb{P})$ such that period n is adapted, predictable employment decision h and effort recommendation a, and monitoring technology $\mathbf{\Gamma}$ which is a martingale with $\mathbf{\Gamma}_0 = 0$. I write the monitoring in period n as $\mathbf{L}_n := \mathbf{\Gamma}_n - \mathbf{\Gamma}_{n-1} + 1$. Note that $\mathcal{F}_0$ represents public randomization in the first period.

The principal's discrete-time problem is to offer a incentive scheme to maximize her value

$$\mathbb{E}^a\left[\sum_{n=1}^{\infty} e^{-(n-1)r\Delta t} h_n\left(\left(1-e^{-r\Delta t}\right)\pi a_n - C\left(\mathbf{L}_n\right)\right)\right]$$

subject to the agent's incentive compatibility constraint

$$a \in \max_{a'} \mathbb{E}^{a'}\left[\sum_{n=1}^{\infty} e^{-(n-1)r\Delta t}\left(1-e^{-r\Delta t}\right) h_n\left(u-ka'_n\right)\right].$$

In the main model, the principal must incentivize effort when employing the agent, i.e., $h_n = 1 \implies a_n = 1$.

Following Spear and Srivastava (1987), I denote by $V_{\Delta t}(W)$ the principal's value as a function of the agent's continuation value

$$W_n := \mathbb{E}^a_{n-1}\left[\sum_{m=n}^{\infty} e^{-(m-n)r\Delta t} h_m\left(1-e^{-r\Delta t}\right)\left(u-ka_m\right)\right].$$

Without loss of optimality, I restrict attention to incentive schemes that do not monitor the agent when the principal does not recommend effort, e.g., $a_n = 0 \implies \mathbf{L}_n = \delta_1$.

A.1.2 Simple upper and lower bound of value function

A first-best incentive scheme is a incentive scheme without the incentive compatibility constraint. It can be shown that the value of first-best incentive schemes is $V_{FB}(W) = \frac{W}{u-k}\pi$ for $W \in [0, u-k]$ and that $V_{\Delta t} \leq V_{FB}$.

I show the existence of discrete-time incentive schemes for all $W_0 \in (0, u-k)$ by constructing the stationary incentive schemes. I restrict to sufficiently short periods such that $u - k > 2(1 - e^{-r\Delta t})u$.

For $W_0 = 0$, define the trivial incentive scheme by the filtration generated by period $\{n\}$, employment decision and recommended effort $h = a \equiv 0$, and no monitoring $\mathbf{\Gamma} \equiv 0$. It is incentive compatible and offers no value to the principal or the agent.

I define the stationary incentive scheme $\mathcal{M}_{ST}$ in which the principal employs the agent, recommends effort, and monitors with a fixed binary experiment until the first bad signal. For $W_0 \in \left[2(1-e^{-r\Delta t})u, u-k\right)$, let the probability of bad signal be $p := \frac{1-e^{-r\Delta t}}{e^{-r\Delta t}} \frac{u-k-W_0}{W_0} \in (0,1)$, the likelihood ratio of bad signal $L_b := 1 + \frac{k}{u-k-W_0}$, and the likelihood ratio of good news $L_g := 1 - \frac{p}{1-p}(L_b - 1)$. Let $(\Omega, \{\mathcal{F}_n\}, \mathbb{P})$ be the augmented filtration generated by a geometrically distributed stopping time τ, the arrival of the bad signal, and period $\{n\}$. The employment decision and effort recommendations are $h_n = a_n := \mathbf{1}_{n \le \tau}$. The monitoring is $\mathbf{\Gamma}_n := \sum_{m=1}^{n} \mathbf{1}_{n<\tau}(L_g - 1) + \mathbf{1}_{n \ge \tau}(L_b - 1)$.

The stationary incentive scheme gives the agent utility W_0 and is incentive compatible due to the specified probability and likelihood ratio of the bad signal. The principal's payoff is $V_{\Delta t}(\mathcal{M}_{ST}) = \frac{(1-e^{-r\Delta t})\pi - (pc(L_b) + (1-p)c(L_g))}{1-(1-p)e^{-r\Delta t}}$. It satisfies $\lim_{W_0 \uparrow u-k} V_{\Delta t}(\mathcal{M}_{ST}) = \pi - \frac{1}{e^{-r\Delta t}} \frac{k}{u-k} c'(\infty)$ and

$$\lim_{W_0 \uparrow u-k} \partial_{W_0} V_{\Delta t}(\mathcal{M}_{ST}) = -\frac{\lim c'(L_b)(L_b - 1) - c(L_b)}{1 - e^{-r\Delta t}} = -\infty \tag{4}$$

where the second equality follows from the Inada condition.

For $W_0 \in [0, 2(1-e^{-r\Delta t})u)$, the principal publicly randomize between the stationary incentive scheme for $2(1-e^{-\Delta t})u$ and the trivial incentive scheme.

A.1.3 Discrete-time value function

Lemma 3 *The value function $V_{\Delta t}$ satisfies $V_{\Delta t}(W) \le \frac{1}{u-k}\left(\pi - c\left(\frac{u-k}{u}\right)\right) W$ for $W \in [0, u-k)$.*

Proof. I prove the lemma by studying a relaxed incentive provision problem that pools all incentive compatibility constraints.

Define the relaxed problem as a static incentive scheme with the following time line.

1. The agent privately chooses effort $a \in \{0, 1\}$.

2. A public randomization is realized.

3. The principal publicly chooses monitoring $\mathbf{L} \in \Delta_1(0,\infty)$ on the private effort.

4. The principal publicly chooses whether to employ the agent $h \in \{0,1\}$.

The agent's vNM payoff is $h(u-ka)$, and the principal's $\pi ha - C(\mathbf{L})$. The problem is parametrized by the agent's continuation value $W_0 \in [0, u-k)$.

The relaxed problem differs from the dynamic incentive provision problem in two ways. First, the agent chooses his private effort at only one information set so that the incentive scheme is static. Second, the principal monitors and acquires the public signal before employing the agent so that incentives are fully backloaded.

For any dynamic incentive scheme with the agent's continuation value W_0 and the principal's value $V_0 > -\infty$, I construct a static incentive scheme that pools all the agent's information sets. The static public randomization draws geometric random variable $n \in \{1,2,...\}$ parametrized by success probability $1-e^{-r\Delta t}$, and a realization of public randomizations from the dynamic scheme. Conditional on n and public randomizations, the principal monitors with the reduced monitoring $\mathbf{L} := \prod_{m=1}^{n-1} \mathbf{L}_m$ from the first $n-1$ monitoring in the dynamic incentive scheme, i.e., the one that gives product likelihood ratio $\prod_{m=1}^{n-1} L_m$ of the first $n-1$ signals in the dynamic incentive scheme. Employment decision h equals h_n corresponding to the sequence $\{L_m : m \leq n-1\}$ in the dynamic scheme.

I compute the agent's expected payoff and show incentive compatibility. For $a=1$, the agent's expected payoff is

$$\begin{aligned}\mathbb{E}^{a=1}\left[h(u-ka)\right] =& \sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}\mathbb{E}^{a=1}\left[h_n\left(u-ka_n\right)|n\right] \\ =&\mathbb{E}^{a=1}\left[\sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}h_n\left(u-ka_n\right)\right].\end{aligned}$$

Note that the payoff in the static incentive scheme equals continuation utility W_0 in the dynamic incentive scheme. The static incentive compatibility constraint is

$$\begin{aligned}
\mathbb{E}^{a=1}\left[h(u-ka)\right] \geq & \mathbb{E}^{a=0}\left[hu\right] \\
\sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}\mathbb{E}^{a=1}\left[h_n\left(u-ka_n\right)|n\right] \geq & \sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}\mathbb{E}^{a=0}\left[uh_n|n\right] \\
\mathbb{E}^{a=1}\left[\sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}h_n\left(u-ka_n\right)\right] \geq & \mathbb{E}^{a=0}\left[\sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}h_n u\right].
\end{aligned}$$

The static incentive constraint is equivalent to a particular dynamic constraint in the dynamic problem—the agent prefers to exert effort whenever recommended, than never exerting any effort. Therefore, the static incentive scheme inherits incentive compatibility from the dynamic incentive scheme.

I continue to show that the static incentive scheme offers the principal weakly higher payoff. In the static incentive scheme, her expected payoff is

$$\begin{aligned}
\mathbb{E}^{a=1}[\pi ha - C(\mathbf{L})] =& \sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}\mathbb{E}^{a=1}\left[\pi h_n a_n - C(\mathbf{L})|n\right] \\
\leq& \sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}\mathbb{E}^{a=1}\left[\pi h_n a_n - \sum_{m=1}^{n-1} C(\mathbf{L}_m)|n\right] \\
=& \sum_{n=1}^{\infty} e^{-(n-1)r\Delta t}\mathbb{E}^{a=1}\left[(1-e^{-r\Delta t})\pi h_n a_n - e^{-r\Delta t}C(\mathbf{L}_n)|n\right] \\
\leq& \sum_{n=1}^{\infty} e^{-(n-1)r\Delta t}\mathbb{E}^{a=1}\left[(1-e^{-r\Delta t})\pi h_n a_n - C(\mathbf{L}_n)|n\right] \\
=& \mathbb{E}^{a=1}\left[\sum_{n=1}^{\infty} e^{-(n-1)r\Delta t}(1-e^{-r\Delta t})\pi h_n a_n - C(\mathbf{L}_n)\right]
\end{aligned}$$

which is the principal's payoff in the dynamic incentive scheme. The first inequality follows from compound reduction (Assumption 2), and the second equality from the summation by parts and that $V_0 > -\infty$ implies the boundary term vanishes $\lim_{n\to\infty} e^{-nr\Delta t}\mathbb{E}^{a=1}\left[\sum_{m=1}^{n} C(\mathbf{L}_m)\right] = 0$.

Finally, I derive the value of the optimal static incentive scheme. Because the monitoring cost is monotone in the Blackwell order, the revelation principle implies that it suffices to consider monitoring with binary support, for $h = 0, 1$, without public randomization. The promise keeping and binding incentive compatibility constraints pin down the optimal static incentive scheme. The optimal monitoring consists of good signal $L_g := \frac{u-k}{u}$ with probability $p_g := \frac{W_0}{u-k}$ and bad signal $L_b := 1 + \frac{p_g}{1-p_g}(1 - L_g)$ with probability $p_b := 1 - p_g$. The principal employs the agent and recommends effort upon the good signal $h(L_g) = a(L_g) = 1$, and does not employ the agent upon the bad signal $h(L_b) = a(L_b) = 0$. The value of the static problem is thus $\frac{W_0}{u-k} - (p_g c(L_g) + p_b c(L_b))$. It is concave with derivative at $W = 0$ given by $\frac{1}{u-k}\left(\pi - c\left(\frac{u-k}{u}\right)\right)$. The lemma then follows from the concavity. ■

For $\Delta t > 0$, I define four functional operators in order to write the Bellman equation for the discrete-time problem. For function $\tilde{v} : \big((1 - e^{-r\Delta t})u, u - k\big) \to \mathbb{R}$, I first define the randomization operator R by

$$Rv(W) := \sup_{\mathbf{J} \in \Delta[0, u-k)} \mathbb{E}\left[\tilde{v}(J)\right] \qquad (5)$$
$$\text{s.t. } \mathbb{E}[J] = W\,.$$

Second, for $v : [0, u - k) \to \mathbb{R}$, I define the working operator A (for $a = 1$) by

$$Av(W) := \sup_{(\mathbf{L},\mathbf{J}) \in \Delta(\mathbb{R} \times [0, u-k))} (1 - e^{-r\Delta t})\pi - \mathbb{E}\left[c(L)\right] + e^{-r\Delta t}\mathbb{E}\left[v(J)\right] \qquad (6)$$
$$\text{s.t. } \begin{cases} \mathbb{E}\left[L\right] & = 1 \\ (1 - e^{-r\Delta t})(u - k) + e^{-r\Delta t}\mathbb{E}[J] & = W \\ e^{-r\Delta t}\mathbb{E}\left[(1 - L)(J - W)\right] & = (1 - e^{-r\Delta t})k \end{cases}$$

where the supremum is taken as $-\infty$ when the constraints are not feasible. I call the constraints in order the Bayesian plausibility constraint, the promise keeping constraint, and the incentive compatibility constraint. Third, I define the suspension operator N (for no employment) by

$$Nv(W) := e^{-r\Delta t} v\left(e^{r\Delta t} W\right)$$

where v is taken as $-\infty$ outside of its domain. Fourth and final, I define the maximum operator by

$$Mv(W) := \max\{Av(W), Nv(W)\}\,. \tag{7}$$

Lemma 4 *The discrete-time value function $V_{\Delta t}$ is the unique bounded, continuous, and concave solution to the functional equation $v = R \circ Mv$.*

Proof. This lemma follows from the standard arguments for the recursive formulation. The operator $R \circ M$ maps bounded, continuous, and concave functions to bounded, continuous, and concave functions. The boundedness follows because, for any $\epsilon > 0$, $Av(W) > -\infty$ for $W \in \big((1 - e^{-r\Delta t})u + \epsilon, u - k\big)$ and $Nv(W) > -\infty$ for $W \in \big[0, e^{-r\Delta t}(u - k)\big)$ and so $Mv(W) > -\infty$ for all W. The continuity follows from the theorem of maximum. The concavity follows because the randomization operator R is the concave envelope operator. In the space of bounded and continuous functions on $[0, u - k)$ with a finite limit at $u - k$ equipped with the supremum norm, the operator R is a weak contraction and M is a contraction with modulus $e^{-r\Delta t}$; therefore, $R \circ M$ is a contraction with modulus $e^{-r\Delta t}$. The operator $R \circ M$ thus admits a unique fixed point by the contraction mapping theorem, and the fixed point inherits boundedness, continuity, and concavity as the limit. Moreover, the fixed point satisfies the transversality condition because it is bounded. Therefore, it equals the value function $V_{\Delta t}$. ■

With abuse of notation, I refer to the maximization problem (6) with the set of feasible control $\Delta\left((0, \infty) \times [0, u - k]\right)$ replaced with $\mathcal{D} \subset \Delta\left(\mathbb{R}^2\right)$, by the maximization problem on $\mathcal{D}$.

Lemma 5 *For any bounded, continuous, and concave v, the maximization problem on $\mathcal{D}_0 := \Delta\left((0, \infty) \times [0, u - k]\right)$, with extension $v(u - k) := \lim_{W \to u-k} v(W)$, admits a maximizer supported on at most four points. Moreover, the maximum is $Av(W)$.*

Proof. The limit $\lim_{W \to u-k} v(W)$ exists because v is concave and bounded.

For $\epsilon > 0$, the maximization problem on $\mathcal{D}_\epsilon := \Delta\left([\epsilon, 1/\epsilon] \times [0, u - k]\right)$ admits a maximizer supported on at most four points by Theorem 2.1 of Winkler (1988). Because v is bounded, the maximum on $\mathcal{D}_\epsilon$ converges to the supremum on $\mathcal{D}_0$ as $\epsilon \to 0$.

I claim that there exists $\epsilon > 0$ such that the maximum on $\mathcal{D}_\epsilon$ equals the supremum on $\mathcal{D}_0$. Suppose, for any $\epsilon > 0$, there exists $\epsilon' \in (0, \epsilon)$ such that the maximum on $\mathcal{D}_{\epsilon'}$ is strictly higher than the maximum on $\mathcal{D}_\epsilon$. Let $\{\epsilon^n\}$ be a decreasing sequence such that $\epsilon^n \to 0$ and the maximum on $\mathcal{D}_{\epsilon^n}$ is strictly increasing. Let $\{(p_i^n, L_i^n, J_i^n) : 1 \leq i \leq 4\}$ be a maximizer for the problem on $\mathcal{D}_{\epsilon^n}$, where p_i^n is the probability of (L_i^n, J_i^n). Without loss of generality, I have $p_i^n > 0$ by possibly splitting some probability mass.

Because $\left(J_i^n\right)_n \in [0, u-k]^4$, it converges to $(J_i) := \lim_n \left(J_i^n\right)$ (along a subsequence). As $\lim_{L\to 0,\infty} c(L) = \infty$ implied by the Inada condition, there exists i such that (p_i^n, L_i^n) also converges with $p_i := \lim_n p_i^n > 0$ and $L_i := \lim_n L_i^n \in (0, \infty)$. I claim that I can choose such i with $J_i \in (0, u-k)$. If not, all such i satisfied $J_i \in \{0, u-k\}$ and so the uniform continuity of v (continuous on compact set $[0, u-k]$) implies $\lim_n \sum_i p_i^n v(J_i^n) = \sum_i p_i v(J_i)$. The maximum on $\mathcal{D}_{\epsilon^n}$ converges to the maximum on $\Delta\left((0,\infty) \times \{0, u-k\}\right)$, which admits a maximizer supported on at most two points. This contradicts the strictly increasing maxima along $\mathcal{D}_{\epsilon^n}$'s. Therefore, I can enumerate $i = 1$ such that $p_1 > 0$, $L_1 \in (0, \infty)$, and $J_1 \in (0, u-k)$.

Along a subsequence, there exists i such that L_i^n diverges to 0 or ∞. Enumerate this as $i = 2$. Because $\lim_{L\to 0,\infty} c(L) = \infty$, the maximizer must satisfy $\lim_n p_2^n = 0$.

I first consider $L_2 \to \infty$. The optimality of the controls implies that a worse signal leads to lower continuation value $J_2^n < J_1^n$. Take $l > 1$. For all n, I define a control $\left\{\left(\tilde{p}_i^n, \tilde{L}_i^n, \tilde{J}_i^n\right) : 0 \leq i \leq 4\right\}$ parametrized by $\tilde{L}_2^n$ as follows. I abbreviate all n dependence for ease of notation. For $i = 3, 4$, define $\tilde{p}_i = p_i$, $\tilde{L}_i = L_i$, and $\tilde{J}_i = J_i$. For $i = 0, 1, 2$, I define $\left\{\left(\tilde{p}_i, \tilde{L}_i, \tilde{J}_i\right)\right\}_i$ parametrized by $\tilde{L}_i$ implicitly by the law of total probability, Bayesian plausibility, promise keeping, binding incentive-compatibility constraints, and $\tilde{L}_0 = lL_1$, $\tilde{J}_0 = J_2$, $\tilde{J}_2 = J_2$, and $\tilde{p}_2\left(1 - \tilde{L}_2\right)\tilde{J}_2 = p_2(1 - L_2)J_2$. Because the constraints are continuously differentiable, the implicit function theorem implies such a control exists in the neighborhood of $(\tilde{p}_0 = 0, \tilde{p}_1 = p_1, \tilde{L}_1 = L_1, \tilde{J}_1 = J_1, \tilde{p}_2 = p_2, \tilde{L}_2 = L_2)$. In that neighborhood, this control as a function of $\tilde{L}_2$ is differentiable with derivatives

$$\begin{aligned}
\frac{d\tilde{p}_0}{d\tilde{L}_2} &= -\frac{d\tilde{p}_1}{d\tilde{L}_2} = \frac{p_2}{1-L_2}\frac{1-L_1}{L_1(1-l)} < 0 \\
\frac{d\tilde{L}_1}{d\tilde{L}_2} &= 0 \\
\frac{d\tilde{J}_1}{d\tilde{L}_2} &= \frac{p_2}{1-L_2}\frac{1}{p_1}\frac{1-lL_1}{L_1(1-l)}(J_1 - J_2) < 0 \\
\frac{d\tilde{p}_2}{d\tilde{L}_2} &= \frac{p_2}{1-L_2} < 0\,.
\end{aligned}$$

Therefore, this control is feasible for sufficiently small $d\tilde{L}_2 = \tilde{L}_2 - L_2 < 0$ and sufficiently small ϵ. At the maximizer, the objective has directional derivative

$$\begin{aligned}
&\frac{d}{d\tilde{L}_2}\left((1-e^{-r\Delta t})\pi - \sum_i \tilde{p}_i c(\tilde{L}_i) + e^{-r\Delta t}\sum_i \tilde{p}_i v(\tilde{J}_i)\right) \\
=&\Bigg(c'(L_2)(L_2-1) - c(L_2) \\
&\quad + c(L_1) + e^{-r\Delta t}\left(v(J_2) - v(J_1)\right) \\
&\quad + \left(c(L_1) - c(lL_1) + e^{-r\Delta t}\left(v(J_2) - v(J_1)\right)\right)\frac{1-L_1}{L_1(1-l)} \\
&\quad + e^{-r\Delta t}(J_1 - J_2)v'(J_1)\frac{1-lL_1}{L_1(1-l)}\Bigg)\frac{p_2}{1-L_2}\,.
\end{aligned}$$

As $n \to \infty$, the Inada condition implies that $c'(L_2)(L_2) - c(L_2)$ diverges to ∞ because $L_1 \to \infty$. Therefore, for some large enough n and then small enough $d\tilde{L}_2 < 0$, this control yields strictly higher objective than the maximizer, a contradiction.

The case of $L_2 \to 0$ is analogous, with $l \in (0,1)$ and $d\tilde{L}_2 > 0$.

Because there exists $\epsilon > 0$ such that the maximum on $\mathcal{D}_\epsilon$ equals the supremum on $\mathcal{D}_0$, the maximization problem on $\mathcal{D}_0$ admits a maximizer supported on at most four points. This maximizer can be approached in value in $\Delta\left((0,\infty) \times [0, u-k)\right)$ so the maximum equals $Av(W)$. ■

Lemma 6 *For any bounded, continuous, and concave v, the function Av is strictly concave.*

Proof. For any $W^- < W^+$, let $\left\{\left(p_i^{\pm}, L_i^{\pm}, J_i^{\pm}\right) : 1 \leq i \leq 4\right\}$ be the respective maximizer on Δ_0 by Lemma 5. For any $\alpha \in (0,1)$, let $W_0 := \alpha W^+ + (1-\alpha)W^-$. It suffices to construct a control at W_0 that yields strictly more than $\alpha Av(W^+) + (1-\alpha)Av(W^-)$.

For $\epsilon > 0$, define $\eta := \frac{W^+ - W^-}{(1-e^{-rdt})k}\epsilon > 0$. Construct a control with eight-point support

$$\Bigg\{ \left(\alpha p_i^+, (1 - \frac{\epsilon}{\alpha}) \left(1 + (1-\eta)\left(L_i^+ - 1\right)\right), J_i^+\right), \\ \left((1-\alpha)p_i^-, (1 + \frac{\epsilon}{1-\alpha}) \left(1 + (1-\eta)\left(L_i^- - 1\right)\right), J_i^-\right) : 1 \leq i \leq 4 \Bigg\}.$$

This control corresponds to the reduced monitoring of a sequence of two compound monitoring. The first monitoring $\left\{\left(\alpha, 1 - \frac{\epsilon}{\alpha}, W^+\right), \left(1-\alpha, 1 + \frac{\epsilon}{1-\alpha}, W^-\right)\right\}$ is a binary monitoring of weak informativeness $L - 1 = O(\epsilon)$. The good signal leads to W^+ and the bad signal leads to W^-. The second monitoring, conditional on the first signal, reduces the informativeness of the maximizers at $W^{\pm}$, i.e., $L-1$ is reduced by factor η. It is straightforward to verify that this control satisfies Bayesian plausibility and promise keeping constraints. It also satisfies the binding incentive compatibility constraint due to the definition of η. The distribution of continuation value J for this control is the convex combination of the maximizers at $W^{\pm}$ and so the expectation $\mathbb{E}[v(J)]$ equals the convex combination of the expectations as well.

It remains to show that the monitoring cost of the constructed control is strictly lower than the convex combination of the monitoring costs at $W^{\pm}$. Because the control corresponds to the reduced monitoring of two compound monitoring, it suffices to consider the total cost of the compound monitoring by compound reduction

(Assumption 2). The total cost of the compound monitoring is

$$
\begin{aligned}
&\alpha c\left(1-\frac{\epsilon}{\alpha}\right)+(1-\alpha)c\left(1+\frac{\epsilon}{1-\alpha}\right)\\
&\quad+\alpha\sum_i p_i^+ c\left(1+(1-\eta)(L_i^+-1)\right)+(1-\alpha)\sum_i p_i^- c\left(1+(1-\eta)(L_i^--1)\right)\\
=&\alpha\sum_i p_i^+\left(c(L_i^+)-\eta c'(L_i^+)(L_i^+-1)\right)+(1-\alpha)\sum_i p_i^-\left(c(L_i^-)-\eta c'(L_i^-)(L_i^--1)\right)+o(\epsilon)\\
=&\alpha\sum_i p_i^+ c(L_i^+)+(1-\alpha)\sum_i p_i^- c(L_i^-)\\
&\quad-\eta\left(\alpha\sum_i p_i^+ c'(L_i^+)(L_i^+-1)+(1-\alpha)\sum_i p_i^- c'(L_i^-)(L_i^--1)\right)+o(\epsilon)\,.
\end{aligned}
$$

The first equality holds because c is differentiable and $c'(1)=0$. The convexity of c implies that $c'(L)(L-1)\geq 0$ with equality only if $L=1$. Because neither L_i^+ or L_i^- is identically one, $\alpha\sum_i p_i^+ c'(L_i^+)(L_i^+-1)+(1-\alpha)\sum_i p_i^- c'(L_i^-)(L_i^--1)>0$ and thus the compound monitoring incur strictly lower monitoring cost than the convex combination of monitoring costs at $W^{\pm}$ for sufficiently small ϵ. ■

The value function $V_{\Delta t}$ may fail to be differentiable but it always admits one-sided derivatives due to its concavity. Whenever I refer to a derivative without specifying the direction, the claim applies to each one-sided derivative.

Lemma 7 *The value function $V_{\Delta t}$ satisfies $\lim_{W\to u-k} V_{\Delta t}(W)=\lim_{W\to u-k} AV_{\Delta t}(W)=\pi-\frac{1}{e^{-r\Delta t}}\frac{k}{u-k}c'(\infty)$ and $\lim_{W\to u-k} V'_{\Delta t}(W)=-\infty$.*

Proof. I prove the lemma in two steps. First, I show $\lim_{W\to u-k}(Av)'(W)=-\infty$ for all bounded, continuous, and concave v. Second, I show that, if bounded and continuous $\tilde{v}$ is concave on $(e^{-r\Delta t}(u-k),u-k)$ and satisfies $\lim_{W\to u-k}\tilde{v}'(W)=-\infty$, then $R\tilde{v}$ satisfies $\lim_{W\to u-k}(R\tilde{v})'(W)=-\infty$. The lemma then follows from the fixed-point property of $V_{\Delta t}$.

To show the first step, I bound the function Av. The objective of a particular control

$$(L,J)=\begin{cases}\left(\frac{u-W}{u-k-W},0\right) & \text{with probability } \frac{1-e^{-r\Delta t}}{e^{-r\Delta t}}\frac{u-k-W}{W}\\ \left(\frac{W-(1-e^{-r\Delta t})u}{W-(1-e^{-r\Delta t})(u-k)},W\right) & \text{with probability } \frac{W-(1-e^{-r\Delta t})(u-k)}{e^{-r\Delta t}W}\end{cases}$$

provides a lower bound for $Av(W)$. As $W\to u-k$, this lower bound converges to $(1-e^{-r\Delta t})\pi-\frac{1-e^{-r\Delta t}}{e^{-r\Delta t}}\frac{k}{u-k}c'(\infty)+e^{-r\Delta t}\lim_{W\to u-k}v(W)$, and its derivative converges to $-\infty$ by the Inada condition.

With $\mathbb{E}\left[v(J)\right]$ replaced by $v\left(\mathbb{E}[J]\right)$, the relaxed maximization problem

$$\sup_{(L,J)\in\Delta(\mathbb{R}\times[0,u-k])}(1-e^{-r\Delta t})\pi-\mathbb{E}\left[c(L)\right]+e^{-r\Delta t}v\left(\mathbb{E}\left[J\right]\right)$$

$$\text{s.t.}\begin{cases}\mathbb{E}\left[L\right] & =1\\ (1-e^{-r\Delta t})(u-k)+e^{-r\Delta t}\mathbb{E}[J] & =W\\ e^{-r\Delta t}\mathbb{E}\left[(1-L)J\right] & =(1-e^{-r\Delta t})k\,,\end{cases}$$

where $v(u-k)$ is defined by continuity, provides an upper bound of Av due to the Jensen inequality because v is concave. As $W\to u-k$, the upper bound converges to $(1-e^{-r\Delta t})\pi-\frac{1-e^{-r\Delta t}}{e^{-r\Delta t}}\frac{k}{u-k}c'(\infty)+e^{-r\Delta t}\lim_{W\to u-k}v(W)$.

Because the lower and upper bound converge to the same limit, Av also converges to that limit by the sandwich theorem. Moreover, the derivative of the lower bound diverges to $-\infty$ and so the derivative of concave function Av also diverges to $-\infty$.

To show the second step, I prove that $R\tilde{v}(W)=\tilde{v}(W)$ for W sufficiently close to $u-k$. Let $\tilde{W}:=\inf\left\{W>\frac{1+e^{-r\Delta t}}{2}(u-k):\tilde{v}'(W)<-\frac{4\sup|\tilde{v}|}{(1-e^{-r\Delta t})(u-k)}\right\}$. For all $W>\tilde{W}$, consider the hyperplane passing through $(W,\tilde{v}(W))$ with slope $\tilde{v}'(W)$. It dominates $\tilde{v}$ on $W>e^{-r\Delta t}(u-k)$ because $\tilde{v}$ is concave there. It dominates $\tilde{v}$ on $W\in[0,e^{-r\Delta t}(u-k)]$ as well because of the upper bound on the slope. Therefore, the randomization $R\tilde{v}(W)$ coincides with $\tilde{v}(W)$, and so $\lim_{W\to u-k}(R\tilde{v})'(W)=\lim_{W\to u-k}\tilde{v}'(W)=-\infty$.

Because $Mv=Av$ on $(e^{-r\Delta t}(u-k),u-k)$, Lemma 8 implies that Mv is concave there. Because $V_{\Delta t}$ is the fixed point of $R\circ M$ (Lemma 4), it follows that

$\lim_{W \to u-k} V'_{\Delta t}(W) = -\infty$. Moreover, the fixed point equation as $W \to u - k$ gives

$$\begin{aligned}\lim_{W \to u-k} V_{\Delta t}(W) =& \lim_{W \to u-k} R \circ MV_{\Delta t}(W) \\ =& \lim_{W \to u-k} AV_{\Delta t}(W) \\ =& (1 - e^{-r\Delta t})\pi - \frac{1 - e^{-r\Delta t}}{e^{-r\Delta t}} \frac{k}{u-k} c'(\infty) + e^{-r\Delta t} \lim_{W \to u-k} V_{\Delta t}(W)\end{aligned}$$

which implies $\lim_{W \to u-k} V_{\Delta t}(W) = \pi - \frac{1}{e^{-r\Delta t}} \frac{k}{u-k} c'(\infty)$. ■

Lemma 8 *The value function $V_{\Delta t}$ is continuous and concave with $V_{\Delta t}(0) = 0$.*

Proof. $V_{\Delta t}$ is concave because R maps concave functions to concave functions. Concavity implies continuity in the interior $(0, u - k)$. Lemma 7 implies $V_{\Delta t}(0) = 0$ and $\limsup_{W \to 0} V_{\Delta t}(0) = 0$. For $W_0 \in (0, u - k)$, the fixed-point equation gives

$$V_{\Delta t}(W_0) = R \circ MV_{\Delta t}(W_0) \geq \frac{u - k - W_0}{u-k} MV_{\Delta t}(0) + \frac{W_0}{u-k} \lim_{W \to u-k} MV_{\Delta t}(W) = \frac{W_0}{u-k} \lim_{W \to u-k} V_{\Delta t}(W)$$

which vanishes as $W_0 \to 0$. Therefore $\liminf_{W \to 0} V_{\Delta t}(W) = 0$. ■

A.1.4 Optimal discrete-time incentive scheme

Lemma 9 *For $W \in [0, u - k)$, the maximization problem of the randomization operator (5) for $\tilde{v} = MV_{\Delta t}$ admits a maximizer supported on at most two points.*

Proof. Consider the maximization problem on $\Delta[0, u-k]$ with the continuous extension $MV_{\Delta t}(u - k) := \pi - \frac{1}{e^{-r\Delta t}} \frac{k}{u-k} c'(\infty)$ (7). Because MV is continuous, the problem admits a maximizer supported on at most two points by Theorem 2.1 of Winkler (1988). Let $\{(p_i, J_i) : i = 1, 2\}$ be the maximizer, where p_i is the probability of J_i. It suffices to show that the maximizer is in $\Delta[0, u - k)$.

Suppose $J_1 = u-k$ and $p_1 > 0$ without loss of generality. The promise keeping constraint implies $J_2 < u-k$ and $p_2 > 0$. For $\epsilon > 0$, define the control $\left\{ \left(\tilde{p}_i, \tilde{J}_i \right) : i = 1, 2 \right\}$ by $\tilde{p}_1 := p_1$, $\tilde{p}_2 := p_2$, $\tilde{J}_1 := u - k - \epsilon$, and $\tilde{J}_2 := J_2 + \frac{p_1}{p_2}\epsilon$. This control decreases

J_1 from $u-k$ by ϵ and increases J_2 to maintain the promise keeping constraint. In excess of the maximizer, this control yields

$$\begin{aligned}
&\tilde{p}_1 V_{\Delta t}(\tilde{J}_1) + \tilde{p}_2 V_{\Delta t}(\tilde{J}_2) - p_1 V_{\Delta t}(J_1) - p_2 V_{\Delta t}(J_2) \\
=& p_1 \left(\frac{V_{\Delta t}(u-k-\epsilon) - V_{\Delta t}(u-k)}{\epsilon} + V'_{\Delta t}(J_2) \right) \epsilon + o(\epsilon) \, .
\end{aligned}$$

Because $\lim_{W \to u-k} V'_{\Delta t}(W) = -\infty$, the control yields strictly higher objective than the maximizer, a contradiction. ■

Lemma 10 *For $W \in ((1-e^{-r\Delta t})u, (u-k))$, the maximization problem of the work operator (6) for $v = V_{\Delta t}$ admits a maximizer supported on at most four points.*

Proof. Lemma 7 and Lemma 8 shows that $V_{\Delta t}$ is bounded, continuous, and concave. Therefore, Lemma 5 implies that a maximizer for $v = V_{\Delta t}$ exists on $\Delta\left((0,\infty) \times [0, u-k]\right)$, and the maximizer is supported on at most four points. Let $\{(p_i, L_i, J_i) : 1 \le i \le 4\}$ with $p_i > 0$ be the maximizer at W. Therefore, it suffices to show this maximizer is on $\Delta\left((0,\infty) \times [0, u-k)\right)$.

Suppose otherwise, i.e. there exists i such that $J_i = u-k$. Because the monitoring cost is monotonic in the Blackwell order, all i's with $J_i = u-k$ have the same L_i. I pool them together and enumerate it as $i = 1$. By the promise keeping constraint, there exists i such that $J_i < u-k$. I enumerate it as $i = 2$.

For $\epsilon > 0$, define the control $\left\{ \left(\tilde{p}_i, \tilde{L}_i, \tilde{J}_i \right)_i \right\}$ by $\left(\tilde{p}_i, \tilde{L}_i, \tilde{J}_i \right) := (p_i, L_i, J_i)$ for $i = 3, 4$, and

$$\begin{aligned}
\tilde{p}_1 :=& p_1, \tilde{p}_2 := p_2 \\
\tilde{L}_1 :=& L_1 - \frac{L_2 - L_1}{J_1 - J_2 - \left(1 + \frac{p_1}{p_2}\right)\epsilon}\epsilon, \tilde{L}_2 := L_2 + \frac{p_1}{p_2} \frac{L_2 - L_1}{J_1 - J_2 - \left(1 + \frac{p_1}{p_2}\right)\epsilon}\epsilon \\
\tilde{J}_1 :=& u - k - \epsilon, \tilde{J}_2 := J_2 + \frac{p_1}{p_2}\epsilon \, .
\end{aligned}$$

This control decreases J_1 from $u-k$ to $u-k-\epsilon$ and adjusts L_1 and (p_2, L_2, J_2) to satisfy the constraints. It is feasible for sufficiently small ϵ.

The change in objective compared to the maximizer is

$$\begin{aligned}&\sum_i \left(\tilde{p}_i V_{\Delta t}(\tilde{J}_i) - p_i V_{\Delta t}(J_i)\right) - \sum_i \left(\tilde{p}_i c(\tilde{L}_i) - p_i c(L_i)\right)\\ =&p_1 \left(\frac{V_{\Delta t}(u-k-\epsilon) - V_{\Delta t}(u-k)}{\epsilon} + V'_{\Delta t}(J_2) - \frac{L_2 - L_1}{J_1 - J_2}\left(c'(L_1) - c'(L_2)\right)\right)\epsilon + o(\epsilon)\,.\end{aligned}$$

Because $\lim_{W \to u-k} V'_{\Delta t}(W) = -\infty$, the control strictly dominates the maximizer for sufficiently small ϵ, a contradiction. ■

Define the cutoff $\underline{W}_{\Delta t} := \max\{W : V_{\Delta t}(W) = V'_{\Delta t}(0)W\}$. It exists because $\lim_{W \to u-k} V'_{\Delta t}(W) = -\infty$ (Lemma 7). I call W an extreme point if $(W, V_{\Delta t}(W))$ is an extreme point of the hypograph of $V_{\Delta t}$. I call an extreme point W interior if $W \neq 0$.

Lemma 11 *The cutoff is bounded by $\underline{W}_{\Delta t} > (1 - e^{-r\Delta t})u$. Promised utility W is an extreme point if and only if $W = 0$ or $W \in [\underline{W}_{\Delta t}, u-k)$. For interior extreme point W, work is strictly optimal over suspension, i.e., $V_{\Delta t}(W) = AV_{\Delta t}(W) > NV_{\Delta t}(W)$.*

Proof. I prove the lemma by showing that work is strictly optimal at W if it is an interior extreme point, and that W is an interior extreme point for all $W \in [\underline{W}_{\Delta t}, u-k)$.

I first show that work is optimal for interior extreme points. Let W be an interior extreme point. The randomization operator R is increasing and so $V_{\Delta t} = RMV_{\Delta t} \geq MV_{\Delta t}$. Because W is an extreme point, I have $V_{\Delta t}(W) = RMV_{\Delta t}(W) = MV_{\Delta t}(W)$. Suppose suspension is optimal $V_{\Delta t}(W) = NV_{\Delta t}(W)$. Because $V_{\Delta t}(0) = 0$ (Lemma 8), the optimality implies

$$V_{\Delta t}(W) = e^{-r\Delta t} V_{\Delta t}\left(e^{r\Delta t}W\right) = (1 - e^{-r\Delta t})V_{\Delta t}(0) + e^{-r\Delta t} V_{\Delta t}\left(e^{r\Delta t}W\right)$$

which contradicts with the fact that W is an interior extreme point. Therefore, $V_{\Delta t}(W) = AV_{\Delta t}(W) > NV_{\Delta t}(W)$.

I continue to show that W is an interior extreme point for all $W \in [\underline{W}_{\Delta t}, u-k)$. By definition, $\underline{W}_{\Delta t}$ is an extreme point. Suppose $\underline{W}_{\Delta t} \leq (1 - e^{-r\Delta t})u$. Then either $\underline{W}_{\Delta t} > 0$ is an interior extreme point or $\underline{W}_{\Delta t} = 0$ and so there exists a sequence of interior extreme points $W_n \to 0$. But working is infeasible and thus suboptimal

$AV_{\Delta t}(W) = -\infty$ for $W \in [0,(1-e^{-r\Delta t})u)$, a contradiction. Therefore, $\underline{W}_{\Delta t} > (1-e^{-r\Delta t})u$ and so it is an interior extreme point.

Because $V_{\Delta t}$ is concave and satisfies $\lim_{W\to u-k} V'_{\Delta t}(W) = -\infty$, there exists a sequence of interior extreme points $W_n \to u-k$.

Therefore, for $W \in (\underline{W}_{\Delta t}, u-k)$, there exists interior extreme points W_1, W_2 such that $W \in (W_1, W_2)$. Then W is an interior extreme point for all $W \in (W_1, W_2)$ due to the strict concavity of $AV_{\Delta t}$ (Lemma 6) and the monotonicity of R. ■

Lemma 12 *Suppose work is optimal $V_{\Delta t}(W) = AV_{\Delta t}(W)$ at W, and $\{(p_i, L_i, J_i) : 1 \le i \le 4\}$ is a solution to the maximization problem (6). Then J_i is an extreme point, i.e. $J_i \in \{0\}\bigcup[\underline{W}_{\Delta t}, u-k)$, whenever $p_i > 0$.*

Proof. Suppose J_1 is not an extreme point and $p_1 > 0$. There exists $J_1^+ > J_1^-$ and $\alpha \in (0,1)$ such that $J_1 = \alpha J_1^+ + (1-\alpha)J_1^-$ and $V_{\Delta t}(J_1) = \alpha V_{\Delta t}(J_1^+) + (1-\alpha)V_{\Delta t}(J_1^-)$.

For $\epsilon > 0$, let $\eta := \frac{p_1 L_1\left(J_1^+ - J_1^-\right)}{\frac{1-e^{-r\Delta t}}{e^{-r\Delta t}}k - \epsilon(1-L_1)\left(J_1^+ - J_1^-\right)}\epsilon > 0$. Define the control $\left\{\left(p_1^{\pm}, L_1^{\pm}, J_1^{\pm}\right), (p_i, \tilde{L}_i, J_i) : 1 \le i\right.$ at W by

$$
\begin{aligned}
p_1^+ :=& \alpha p_1, \quad p_2^- := (1-\alpha)p_1 \\
L_1^+ :=& \left(1 - \frac{\epsilon}{\alpha}\right)\left(1 + (1-\eta)(L_1 - 1)\right), \quad L_1^- := \left(1 + \frac{\epsilon}{1-\alpha}\right)\left(1 + (1-\eta)(L_1 - 1)\right) \\
\tilde{L}_i :=& 1 + (1-\eta)(L_i - 1) \quad \text{ for } i = 2,3,4\,.
\end{aligned}
$$

This control corresponds to the reduced monitoring of two compound monitoring. The first monitoring $\{(p_i, 1 + (1-\eta)(L_i - 1), J_i) : 1 \le i \le 4\}$ is a less informative version of the monitoring in the maximizer. The informativeness of each signal is reduced by factor η. The continuation value jumps to J_i for each i as in the maximizer. The second contingent monitoring is informative only upon the first signal $i = 1$ in the first monitoring; it is the degenerate monitoring for the other signals $i = 2,3,4$. Upon the first signal, it is a binary monitoring $\left\{\left(\alpha, 1 - \frac{\epsilon}{\alpha}, J_1^+\right), \left(1-\alpha, 1 + \frac{\epsilon}{1-\alpha}, J_1^-\right)\right\}$ of weak informativeness $L - 1 = O(\epsilon)$. The good signal leads to J^+ and the bad signal leads to J^-. It is straightforward to verify that this control satisfies Bayesian plausibility and promise keeping constraints. It also satisfies the binding incentive constraint due to the definition of η. Because $V_{\Delta t}(J_1) = \alpha V_{\Delta t}(J_1^+) + (1-\alpha)V_{\Delta t}(J_1^-)$, this control gives the same expectation $\mathbb{E}\left[V_{\Delta t}(J)\right]$ as the maximizer.

It remains to show that the monitoring cost of the constructed control is strictly lower than the maximizer. Because the control corresponds to a reduced monitoring, it suffices to consider the total cost of the compound monitoring by compound reduction (Assumption 2). The total cost is

$$\begin{aligned}
&\sum_i p_i c\left(1+(1-\eta)(L_i-1)\right) + p_1\left(\alpha^+ c\left(1-\epsilon/\alpha^+\right) + \alpha^- c\left(1+\epsilon/\alpha^-\right)\right) \\
=&\sum_i p_i\left(c(L_i) - \eta c'(L_i)(L_i-1)\right) + o(\epsilon) \\
=&\sum_i p_i c(L_i) - \eta\sum_i p_i c'(L_i)(L_i-1) + o(\epsilon)\,.
\end{aligned}$$

The first equality holds because c is differentiable and $c'(1)=0$. The convexity of c implies that $c'(L)(L-1)\geq 0$ with equality only if $L=1$. Because L_i is not identically one, $\sum_i p_i c'(L_i)(L_i-1) > 0$ and thus the compound monitoring incurs strictly lower monitoring cost than the optimal control for sufficiently small ϵ. This contradicts the optimality of the maximizer. ■

For $\Delta t > 0$ and $W_0 > \underline{W}_{\Delta t}$, I define the canonical discrete-time incentive scheme at continuation value W_0 iteratively. Let $\mathcal{F}_0$ be the trivial measure space. For $n \geq 1$, take an maximizer $\{(p_i, L_i, J_i) : 1 \leq i \leq 4\}$ of $AV_{\Delta t}(W_{n-1})$. Define $(\mathcal{F}_n, \mathbb{P}_n)$ as the product probability space of $(\mathcal{F}_{n-1}, \mathbb{P}_{n-1})$ augmented with n. Define the random variable (L_n, W_n) according to the law of the maximizer (L_i, J_i). Define $h_n \equiv a_n := \mathbf{1}_{W_{n-1}\neq 0}$. The complete probability space $(\Omega, \{\mathcal{F}_n\}, \mathbb{P})$ exists by the Kolmogorov extension theorem.

Lemma 13 *The canonical incentive scheme is optimal.*

Proof. It follows from Lemma 11 and Lemma 12 by the standard recursive argument. ■

Let $N := \inf\{n : W_n = 0\} \in \mathbb{N}\bigcup\{\infty\}$. It follows from Lemma 12 that $h_n \equiv a_n = \mathbf{1}_{n\leq N}$.

A.1.5 Uniform convergence and limit

Lemma 14 $\lim_{\Delta t \to 0} \underline{W}_{\Delta t} = 0$.

Proof. Suppose otherwise, i.e. there exists a sequence $\Delta t \to 0$ such that $\underline{W}_{\Delta t} \geq \underline{W} \in (0, u-k)$. I shall construct a incentive scheme that attains straightly higher value for sufficiently small $\Delta t > 0$.

Consider $\{(p_i, L_i, J_i) : i = 1, 2\}$ defined by $J_1 := 0$, $J_2 := \underline{W}$, and

$$\begin{aligned}
p_1 :=& \frac{W_0 - (1-e^{-r\Delta t})(u-k) - e^{-r\Delta t} J_2}{e^{-r\Delta t}(J_1 - J_2} \\
p_2 :=& \frac{W_0 - (1-e^{-r\Delta t})(u-k) - e^{-r\Delta t} J_1}{e^{-r\Delta t}(J_2 - J_1} \\
L_1 :=& 1 - \frac{1}{p_1} \frac{1-e^{-r\Delta t}}{e^{-r\Delta t}} \frac{k}{J_1 - J_2} \\
L_2 :=& 1 - \frac{1}{p_2} \frac{1-e^{-r\Delta t}}{e^{-r\Delta t}} \frac{k}{J_2 - J_1} .
\end{aligned}$$

It can be verified that this control satisfies Bayesian plausibility, promise keeping, and binding incentive compatibility constraints.

The value of this control in excess of $V_{\Delta t}(W_0)$ is

$$\begin{aligned}
&(1-e^{-r\Delta t})\pi - p_1 c(L_1) - p_2 c(L_2) + e^{-r\Delta t}\left(p_1 V_{\Delta t}(J_1) + p_2 V_{\Delta t}(J_2)\right) - V_{\Delta t}(W_0) \\
=&(1-e^{-r\Delta t})\pi + e^{-r\Delta t}\left(p_1 V'_{\Delta t}(0) J_1 + p_2 V'_{\Delta t}(0) J_2\right) - V'_{\Delta t}(0) W_0 + o(1-e^{-r\Delta t}) \\
=&(1-e^{-r\Delta t})\left(\frac{\pi}{u-k} - V'_{\Delta t}(0)\right) + o(1-e^{-r\Delta t}) .
\end{aligned}$$

The first equality follows from $1 - L_i = O(1-e^{-r\Delta t})$ and $c'(1) = 0$, and the second from direct computation. Because $V'_{\Delta t}(0) < \frac{\pi}{u-k}$ (Lemma 3), this control attains strictly higher value for sufficiently small Δt, contradicting the optimality of $V_{\Delta t}$. ■

Lemma 15 *The value increases when the period length shrinks, i.e.* $\Delta t > \Delta t' > 0$ *implies* $V_{\Delta t'} \geq V_{\Delta t}$.

Proof. To distinguish the operators for Δt and $\Delta t'$, I write explicitly $A_{\Delta t}$ and $A_{\Delta t'}$

for the operator A defined in Equation (6), and similarly for N and M. Note that the randomization operator R does not depend on the period length.

Because R and $M_{\Delta t'}$ are increasing operators and $V_{\Delta t'}$ is the fixed point of $RM_{\Delta t'}$, it suffices to show that $RM_{\Delta t'}V_{\Delta t} \geq RM_{\Delta t}V_{\Delta t}$.

I first show $N_{\Delta t'}V_{\Delta t}(W) \geq N_{\Delta t}V_{\Delta t}(W)$, or equivalently

$$(1 - e^{-r\Delta t'})V_{\Delta t}(0) + e^{-r\Delta t'}V_{\Delta t}(e^{r\Delta t'}W) \geq (1 - e^{-r\Delta t})V_{\Delta t}(0) + e^{-r\Delta t}V_{\Delta t}(e^{r\Delta t}W)$$

because $V_{\Delta t}(0) = 0$. Observe that $(1 - e^{-r\Delta t'})\delta_0 + e^{-r\Delta t'}\delta_{e^{r\Delta t'}W}$ is a mean-preserving contraction of $(1 - e^{-r\Delta t})\delta_0 + e^{-r\Delta t}\delta_{e^{r\Delta t}W}$, where δ is the Dirac delta. Therefore, the statement follows from the concavity of $V_{\Delta t}$.

I first show $RM_{\Delta t'}V_{\Delta t}(W) \geq RM_{\Delta t}V_{\Delta t}(W)$ for $W \in [\underline{W}_{\Delta t}, u - k)$. Lemma 11 and Lemma 12 implies $RM_{\Delta t}(W) = A_{\Delta t}V_{\Delta t}(W)$. Because R and max are increasing, it suffices to show $A_{\Delta t'}V_{\Delta t}(W) \geq A_{\Delta t}V_{\Delta t}$. Let $\{(p_i, L_i, J_i) : 1 \leq i \leq 4\}$ be a maximizer of $A_{\Delta t}$ at W (Lemma 10). Define the $A_{\Delta t'}$ control $\{(p_0, L_0 := 1, J_0 := W), ((1 - p_0)p_i, L_i, J_i) : 1 \leq i \leq 4\}$ where $p_0 := 1 - e^{-r(\Delta t - \Delta t')}\frac{1-e^{-r\Delta t'}}{1-e^{-r\Delta t}} \in (0, 1)$. It is straightforward to verify that it satisfies the Bayesian plausibility, promise keeping, and binding incentive compatibility constraints of $\Delta t'$ by the Δt versions and the definition of p_0. It attains value

$$\begin{aligned}
&(1 - e^{-r\Delta t'})\pi - p_0 c(L_0) - (1 - p_0)\sum_i p_i V_{\Delta t}(J_i) + e^{-r\Delta t'}\left(p_0 V_{\Delta t}(W) + (1 - p_0)\sum_i p_i V_{\Delta t}(J_i)\right) \\
=&(1 - p_0)e^{-r(\Delta t' - \Delta t)}\left(\frac{1 - e^{-r\Delta t'}}{1 - p_0}e^{-r(\Delta t - \Delta t')}\pi - e^{-r(\Delta t - \Delta t')}\sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i V_{\Delta t}(J_i)\right) \\
&+ e^{-r\Delta t'}p_0 V_{\Delta t}(W) \\
\geq&(1 - p_0)e^{-r(\Delta t' - \Delta t)}\left((1 - e^{-r\Delta t})\pi - \sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i p_i V_{\Delta t}(J_i)\right) + e^{-r\Delta t'}p_0 V_{\Delta t}(W) \\
=&(1 - p_0)e^{-r(\Delta t' - \Delta t)}V(W) + e^{-r\Delta t'}p_0 V_{\Delta t}(W) \\
=&V_{\Delta t}(W)\,.
\end{aligned}$$

The inequality follows from the definition of p_0 and $e^{-r(\Delta t - \Delta t')} < 1$. The second equality follows from the optimality of the maximizer at Δt.

I continue to show $RM_{\Delta t'}V_{\Delta t}(W) \geq RM_{\Delta t}V_{\Delta t}(W)$ for $W \in [0, \underline{W}_{\Delta t})$. Note that $RM_{\Delta t}V_{\Delta t}(W) = V_{\Delta t}(W) = \frac{\underline{W}_{\Delta t}-W}{\underline{W}_{\Delta t}}V_{\Delta t}(0) + \frac{W}{\underline{W}_{\Delta t}}V_{\Delta t}(\underline{W}_{\Delta t})$. Therefore, I have

$$\begin{aligned}
RM_{\Delta t'}V_{\Delta t}(W) \geq & \frac{\underline{W}_{\Delta t} - W}{\underline{W}_{\Delta t}} M_{\Delta t'}V_{\Delta t}(0) + \frac{W}{\underline{W}_{\Delta t}} M_{\Delta t'}V_{\Delta t}(\underline{W}_{\Delta t}) \\
\geq & \frac{\underline{W}_{\Delta t} - W}{\underline{W}_{\Delta t}} N_{\Delta t'}V_{\Delta t}(0) + \frac{W}{\underline{W}_{\Delta t}} A_{\Delta t'}V_{\Delta t}(\underline{W}_{\Delta t}) \\
\geq & \frac{W}{\underline{W}_{\Delta t}} V_{\Delta t}(\underline{W}_{\Delta t}) = V_{\Delta t}(W)\,.
\end{aligned}$$

The first inequality follows because this is a particular randomization, the second from the definition of $M_{\Delta t'}$ as the maximum, and the third from $N_{\Delta t'}V_{\Delta t}(0) = 0$ and $A_{\Delta t'}V_{\Delta t} \geq A_{\Delta t}V_{\Delta t}$. The last equality follows from the definition of $\underline{W}_{\Delta t}$. ■

Corollary 1 *There exists continuous and concave function V_{DL} such that $V_{\Delta t} \to V_{DL}$ point-wise as $\Delta t \to 0$.*

Proof. Because $V_{\Delta t}(W)$ is monotonic in Δt and bounded from above by $\frac{\pi}{u-k}W$, it converges to a limit $V_{DL}(W)$. The limit V_{DL} inherits concavity from $V_{\Delta t}$, and is thus continuous in the interior. It is also continuous at $W = 0$ because $\frac{\pi}{u-k}W \to 0$ as $W \to 0$. ■

For $\Delta t > 0$, take an Δt-optimal canonical incentive scheme. Denote the filtered probability space by $(\Omega, \{\mathcal{F}_n\}, \mathbb{P})$, employment decision by h, effort recommendation by a, and monitoring technology by $\{\mathbf{L}_n\}$.

A.2 Recursive formulation via Poisson incentive schemes

In this subsection, I establish a recursive formulation of the continuous-time incentive provision problem by replicating discrete-time incentive schemes by compound Poisson incentive schemes. The key results are Lemma 20 and Theorem 1. Lemma 20 strengthens Lemma 1. It states that compound Poisson incentive schemes are sufficient to maximize the principal's value. Theorem 1 makes use of the sufficiency to establish an HJB equation of Poisson monitoring.

A.2.1 Compound Poisson incentive schemes

Definition 1 *A tuple* $(\Omega, \mathbb{F}, \mathbb{P}, \Gamma, h, a)$ *is a compound Poisson incentive scheme if there exist discrete-time monitoring technology* $\mathbb{L} := \{\mathbf{L}_n : \operatorname{supp}|\mathbf{L}_n| \leq 4, n = 1, 2, ...\}$ *with* $\mathbf{L}_0 := 1$*,* $\mathbb{L}$*-stopping time* N*, and arrival times* $\{\tau_n : n = 0, 1, 2, ...\}$ *with* $\tau_0 := 0$ *of an independent Poisson process of frequency* $\lambda > 0$ *such that*

- *the cumulative excess likelihood ratio is a compound Poisson process*

$$\mathbf{\Gamma}_t = \sum_{n=1}^{N_t \wedge N} (L_n - 1)$$

  where $N_t := \inf\{n \geq 0 : \tau_n \leq t\}$*;*

- *the filtration* $(\Omega, \mathbb{F}, \mathbb{P})$ *is the augmented natural filtration of* $(t, \mathbf{\Gamma})$*;*

- *the employment decision and effort recommendation follow a cutoff* $h_t = a_t = \mathbf{1}_{t \leq \tau_N}$ *at* τ_N*;*

- *the agent's continuation value* W *satisfies the* ***instantaneous*** *incentive compatibility constraint for* $n \leq N - 1$

$$\lambda \mathbb{E}_{\tau_n} \left[\left(\mathbf{\Gamma}_{\tau_n} - \mathbf{\Gamma}_{\tau_{n+1}} \right) \left(W_{\tau_{n+1}} - W_{\tau_n} \right) \right] = rk \,.$$

Note that $\mathbf{\Gamma}$ is a càdlàg local martingale, and h and a are left-continuous and thus predictable. Moreover, $\mathbf{\Gamma}$ is a martingale if $\mathbb{E}\left[\sum_{n=1}^{\infty} e^{-(n-1)\frac{r}{\lambda}} C(\mathbf{L}_n)\right] < \infty$ because $c'(\infty) < \infty$.

Denote $\Delta W_{\tau_{n+1}} := W_{\tau_{n+1}} - W_{\tau_n}$ and $\Delta\Gamma_{\tau_{n+1}} := \Gamma_{\tau_{n+1}} - \Gamma_{\tau_n}$.

Lemma 16 *A compound Poisson incentive scheme satisfies the* ***intantaneous*** *promise keeping constraint for* $n \leq N - 1$

$$\lambda \mathbb{E}_{\tau_n} \left[W_{\tau_{n+1}} - W_{\tau_n} \right] = r(W_{\tau_n} - u + k) \,.$$

Proof. The law of iterated expectations yields

$$W_{\tau_n} = \mathbb{E}_{\tau_n}\left[\int_{\tau_n}^{\tau_{n+1}} re^{-r(t-\tau_n)}(u-k)dt + e^{-r(\tau_{n+1}-\tau_n)}W_{\tau_{n+1}}\right].$$

Subtracting W_{τ_n} from both sides, I obtain

$$\begin{aligned} 0 =& \mathbb{E}_{\tau_n}\left[\int_{\tau_n}^{\tau_{n+1}} re^{-r(t-\tau_n)}(u-k-W_{\tau_n})dt + e^{-r(\tau_{n+1}-\tau_n)}\Delta W_{\tau_{n+1}}\right] \\ =& \int_{\tau_n}^{\infty} re^{-(\lambda+r)(t-\tau_n)}(u-k-W_{\tau_n})dt + \int_{\tau_n}^{\infty} \lambda e^{-(\lambda+r)(t-\tau_n)}\mathbb{E}_{\tau_n}\left[\Delta W_{\tau_{n+1}}\right]dt \\ =& \frac{r}{\lambda+r}(u-k-W_{\tau_n}) + \frac{\lambda}{r+\lambda}\mathbb{E}_{\tau_n}\left[\Delta W_{\tau_{n+1}}\right] \end{aligned}$$

where the second equality follows because τ_{n+1} arrives with frequency λ independent of $W_{\tau_{n+1}}$. ■

I show that instantaneous incentive compatibility implies incentive compatibility.

Lemma 17 *A compound Poisson incentive scheme is incentive compatible.*

Proof. I first compute the Radon-Nikodym derivative of the change of measure for any predictable a'

$$\left.\frac{d\mathbb{P}^{a'}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t^{a'} = \mathcal{E}\left(\int_0^t (1-a'_s)d\Gamma_s\right) = \prod_{n=1}^{N_t \wedge N}\left(1+\left(1-a'_{\tau_n}\right)\Delta\Gamma_{\tau_n}\right)$$

where $\mathcal{E}$ is the stochastic exponential function and $\Delta\Gamma_s := \Gamma_s - \Gamma_{s-}$ is the jump in Γ at s. The last equality follows because Γ and thus $Z^{a'}$ are pure jump processes.

I show that the agent's continuation value is independent of his effort choice a'. For $n < N$, the Itô lemma gives[14]

$$\mathbb{E}_{\tau_n}^{a'}\left[e^{-r(\tau_{n+1}-\tau_n)}W_t\right] = W_{\tau_n} + \mathbb{E}_{\tau_n}^{a'}\left[\int_{\tau_n}^{\tau_{n+1}} -re^{-r(s-\tau_n)}W_{s-}ds + e^{-r(\tau_{n+1}-\tau_n)}\Delta W_{\tau_{n+1}}\right].$$

[14]The integral $\int_x^y$ integrates over $(x,y]$.

Adding the flow payoffs, I obtain

$$
\begin{aligned}
&\mathbb{E}^{a'}_{\tau_n}\left[\int_{\tau_n}^{\tau_{n+1}} re^{-r(s-\tau_n)}h_s(u-ka'_s)ds + e^{-r(\tau_{n+1}-\tau_n)}W_{\tau_{n+1}}\right]\\
=&W_{\tau_n} + \mathbb{E}^{a'}_{\tau_n}\left[\int_{\tau_n}^{\tau_{n+1}} re^{-r(s-\tau_n)}\left(u-ka'_s-W_{s-}\right)ds + e^{-r(\tau_{n+1}-\tau_n)}\Delta W_{\tau_{n+1}}\right]\\
=&W_{\tau_n} + \mathbb{E}^{a}_{\tau_n}\Bigg[\int_{\tau_n}^{\tau_{n+1}} re^{-r(s-\tau_n)}\left(u-ka'_s-W_{s-}\right)ds\\
&\qquad + e^{-r(\tau_{n+1}-\tau_n)}\Delta W_{\tau_{n+1}}\left(1+\left(1-a'_{\tau_{n+1}}\right)\Delta\Gamma_{\tau_{n+1}}\right)\Bigg]\\
=&W_{\tau_n} + \mathbb{E}^{a}_{\tau_n}\Bigg[\int_{\tau_n}^{\tau_{n+1}} re^{-r(s-\tau_n)}\left(u-ka'_s-W_{s-}\right)ds\\
&\qquad + \mathbb{E}^{a}_{\tau_{n+1}-}\left[e^{-r(\tau_{n+1}-\tau_n)}\Delta W_{\tau_{n+1}}\left(1+\left(1-a'_{\tau_{n+1}}\right)\Delta\Gamma_{\tau_{n+1}}\right)\right]\Bigg]\\
=&W_{\tau_n} + \mathbb{E}^{a}_{\tau_n}\left[\int_{\tau_n}^{\tau_{n+1}} re^{-r(s-\tau_n)}\left(u-ka'_s-W_{s-}\right)ds + e^{-r(\tau_{n+1}-\tau_n)}\left(\mathbb{E}^{a}_{\tau_n}\left[\Delta W_{\tau_{n+1}}\right]-\frac{r}{\lambda}k\mathbf{1}_{a_{\tau_{n+1}}=0}\right)\right]\\
=&W_{\tau_n} + \mathbb{E}^{a}_{\tau_n}\left[\int_0^{\infty} e^{-(\lambda+r)(s-\tau_n)}\left(r\left(u-ka'_s-W_{s-}\right)+\lambda\mathbb{E}_{\tau_n}\left[\Delta W_{\tau_{n+1}}\right]-rk\mathbf{1}_{a'_s=0}\right)ds\right]\\
=&W_{\tau_n} + \mathbb{E}^{a}_{\tau_n}\left[\int_0^{\infty} e^{-(\lambda+r)(s-\tau_n)}\left(r\left(u-k-W_{s-}\right)+\lambda\mathbb{E}_{\tau_n}\left[\Delta W_{\tau_{n+1}}\right]\right)ds\right]\\
=&W_{\tau_n}\,.
\end{aligned}
$$

The first equality follows from the Itô lemma, the second from the change of measure, the third from the law of iterated expectations, the fourth from the instantaneous incentive compatibility constraint, the fifth from the arrival rate of τ_{n+1} being λ, the sixth from algebra, and the seventh from the instantaneous promise keeping constraint (Lemma 16).

From time 0, the agent's continuation value is

$$
\begin{aligned}
&\mathbb{E}^{a'}_{0}\left[\int_0^{t\wedge\tau_N} re^{-rs}h_s(u-ka'_s)ds + e^{-r(t\wedge\tau)}W_{t\wedge\tau_N}\right]\\
=&W_0 + \mathbb{E}^{a'}_0\left[\sum_{n=0}^{N_t\wedge N-1} e^{-r\tau_n}\mathbb{E}^{a'}_{\tau_n}\left[-W_{\tau_n}+\int_{\tau_n}^{\tau_{n+1}} re^{-r(s-\tau_n)}h_s(u-ka'_s)ds + e^{-r(\tau_{n+1}-\tau_n)}W_{\tau_{n+1}}\right]\right]\\
=&W_0\,.
\end{aligned}
$$

The first equality follows from algebra and the second from applying the previous deviation iteratively. Because the agent gets the same payoff for any effort plan a', the recommended effort a is incentive compatible. ■

Lemma 18 *The monitoring cost of a compound Poisson incentive scheme is*

$$C_t(\mathbf{\Gamma}) = \sum_{n=1}^{N_t} C\left(1 + \Delta\mathbf{\Gamma}_{\tau_n}\right) = \sum_{n=1}^{N_t} \mathbb{E}_{\tau_{n-1}}\left[c\left(1 + \Delta\Gamma_{\tau_n}\right)\right] .$$

Proof. For partition $\mathcal{P}$ with $\|\mathcal{P}\| < \frac{1}{2}\max_{n \leq N_t}(\tau_n - \tau_{n-1})$, the sum

$$\begin{aligned}\sum_{(t_m,t_{m+1}]\in\mathcal{P}} C\left(1 + \mathbf{\Gamma}_{t_{m+1}} - \mathbf{\Gamma}_{t_m}\right) &= \sum_{\substack{(t_m,t_{m+1}]\in\mathcal{P} \\ \tau_n\in(t_m,t_{+1}]:n\leq N_t}} C\left(1 + \mathbf{\Gamma}_{t_{m+1}} - \mathbf{\Gamma}_{t_m}\right) \\ &= \sum_{n=1}^{N_t} C\left(1 + \mathbf{\Gamma}_{t_{m+1}} - \mathbf{\Gamma}_{t_m}\right) \\ &= \sum_{n=1}^{N_t} \mathbb{E}_{\tau_{n-1}}\left[c\left(1 + \Delta\Gamma_{\tau_n}\right)\right] .\end{aligned}$$

The first equality follows because the pure jump process satisfies $\Gamma_{t_{m+1}} - \Gamma_{t_m} = 0$ if $(t_n, t_{n+1}] \bigcap \{\tau_n\} = \emptyset$. The second equality follows because $(t_n, t_{n+1}] \bigcap \{\tau_n\}$ has at most one element from the mesh of the partition. The third equality follows from likelihood ratio separability (Assumption 1). ■

Lemma 19 *The value of a compound Poisson incentive scheme is*

$$\mathbb{E}\left[\sum_{n=1}^{N} \left(\frac{\lambda}{\lambda + r}\right)^{n-1} \frac{r}{\lambda + r}\pi - \left(\frac{\lambda}{\lambda + r}\right)^{n} C(\mathbf{L}_n)\right] .$$

Moreover, a compound Poisson incentive scheme can be approximated by discrete-time schemes.

Proof. The value follows from Lemma 18 and direct computation.

I show the approximation from discrete time by construction. Define $\Delta t := \frac{1}{r}\log\frac{\lambda+r}{r} > 0$ so that $e^{-r\Delta t} = \frac{\lambda}{\lambda+r}$ and $1 - e^{-r\Delta t} = \frac{r}{\lambda+r}$. Consider the Δt-canonical incentive scheme defined by $\{\mathbf{L}_n\}$ and N. It satisfies the one-shot deviation principle by the definition of Δt, and it is thus incentive compatible. Its value is

$$\mathbb{E}\left[\sum_{n=1}^{N}\left(\frac{\lambda}{\lambda+r}\right)^{n-1}\frac{r}{\lambda+r}\pi-\left(\frac{\lambda}{\lambda+r}\right)^{n-1}C(\mathbf{L}_n)\right].$$

For $\Delta t' \in (0,\Delta t)$, I construct a $\Delta t'$-incentive scheme with value

$$\mathbb{E}\left[\sum_{n=1}^{N}\left(\frac{\lambda}{\lambda+r}\right)^{n-1}\frac{r}{\lambda+r}\pi-e^{-r(\Delta t-\Delta t')}\left(\frac{\lambda}{\lambda+r}\right)^{n-1}C(\mathbf{L}_n)\right].$$

As $\Delta t' \to 0$, the value of the $\Delta t'$-scheme converges to the compound Poisson one, establishing the approximation.

Let $p_0 := \frac{1-e^{-r(\Delta t-\Delta t')}}{1-e^{-r\Delta t}} \in (0,1)$. Let $\{\tau_m : m \in \mathbb{N}\}$ be a discrete-time Poisson process with success probability $1-p_0$ where $\tau_0 := 0$. Let $N_n := \max\{m : \tau_m \le n\}$ be the counting process and $N_{n-} := \max\{m : \tau_m \le n-1\}$ be the predictable version. I denote by $\mathbb{P}'$ the product probability space of $\{\tau\}$ and $\mathbb{P}$, and $\{\mathcal{F}'_n\}$ the product measure space of $\{\tau\}$ and $\{\mathcal{F}_n\}$ augmented with period $\{n\}$. Define $h'_n := h_{N_{n-}}$ and $a'_n := a_{N_{n-}}$. Define $\mathbf{\Gamma}'_n := \mathbf{\Gamma}_{N_n}$ which is a martingale.

I first show $W'_n = W_{N_{n-}}$ for all n.

$$\begin{aligned}
W'_n =&\mathbb{E}'_{n-1}\left[\sum_{m=n}^{\infty}(1-e^{-r\Delta t'})e^{-r(m-n)\Delta t'}h'_m(u-ka'_m)\right]\\
=&\mathbb{E}'_{n-1}\left[\sum_{j=N_{n-}}^{\infty}\sum_{l=\tau_{j-1}+1}^{\tau_j}(1-e^{-r\Delta t'})e^{-r(l-n)\Delta t'}h'_l(u-ka'_l)\right]\\
=&\mathbb{E}'_{n-1}\left[\sum_{j=N_{n-}}^{N}(1-e^{-r\Delta t})e^{-r(j-N_{n-})\Delta t}h_j(u-ka_j)\right]\\
=&\mathbb{E}_{N_{n-}-1}\left[\sum_{j=N_{n-}}^{N}(1-e^{-r\Delta t})e^{-r(j-N_{n-})\Delta t}h_j(u-ka_j)\right]\\
=&W_{N_{n-}}.
\end{aligned}$$

In the second equality, I partition m by $\{\tau_k\}$. The third equality follows from the definition of p_0. The fourth equality follows from the product structure of $\mathbb{P}'$.

I show that the $\Delta t'$-incentive scheme satisfies the one-shot deviation principle and thus incentive compatibility. The one-shot IC in $\Delta t'$

$$\begin{aligned} e^{-r\Delta t'}\mathbb{E}'_{n-1}\left[(1-L'_n)(W'_{n+1}-W'_n)\right] =&(1-e^{-r\Delta t'})k \\ e^{-r\Delta t'}\left(p_0\times 0+(1-p_0)\mathbb{E}_{N_{n-}-1}\left[(1-L_{N_{n-}})(W_{N_{n-}+1}-W_{N_{n-}})\right]\right) =&(1-e^{-r\Delta t'})k \end{aligned}$$

follows from the one-shot IC in Δt

$$e^{-r\Delta t}\mathbb{E}_{N_{n-}-1}\left[(1-L_{N_{n-}})(W_{N_{n-}+1}-W_{N_{n-}})\right] =(1-e^{-r\Delta t})k$$

due to the definition of p_0.

Finally, I show that the constructed incentive scheme attains the specified value

$$\begin{aligned} &\mathbb{E}'\left[\sum_{n=1}^{\infty}e^{-r(n-1)\Delta t'}h'_n\left((1-e^{-r\Delta t'})\pi a'_n-C(\mathbf{L}'_n)\right)\right] \\ =&\mathbb{E}'\left[\sum_{m=1}^{N}\left(\sum_{n=\tau_{m-1}+1}^{\tau_m}e^{-r(n-1)\Delta t'}(1-e^{-r\Delta t'})\pi-e^{-r(\tau_m-1)\Delta t'}C(\mathbf{L}'_n)\right)\right] \\ =&\mathbb{E}\left[\sum_{m=1}^{N}e^{-r(m-1)\Delta t}\left((1-e^{-r\Delta t})\pi-e^{-r(\Delta t-\Delta t')}C(\mathbf{L}_m)\right)\right] \\ =&\mathbb{E}\left[\sum_{m=1}^{N}\left(\frac{\lambda}{\lambda+r}\right)^{m-1}\frac{r}{\lambda+r}\pi-e^{-r(\Delta t-\Delta t')}\left(\frac{\lambda}{\lambda+r}\right)^{m-1}C(\mathbf{L}_m)\right]. \end{aligned}$$

In the first equality, I partition period n by $\{\tau_m\}$ and invoke $C(\mathbf{L}'_n)=0$ for $\tau_m-1<n<\tau_m$. The second equality follows from the definition of p_0 and $\mathbf{L}'_n$. The third equality follows from the definition of Δt. ■

A.2.2 Proof of Lemma 1

I show in fact a stronger version of Lemma 1. Let V_{CP} be the value function of compound Poisson monitoring schemes. Recall that V is the value function of the continuous-time problem and V_{DL} is the limit value function of discrete-time problems.

Lemma 20 $V_{DL} = V = V_{CP}$.

Proof. I prove the lemma by showing $V_{DL} \geq V$, $V \geq V_{CP}$, and $V_{CP} \geq V_{DL}$. The first inequality, $V_{DL} \geq V$, follows because each continuous-time incentive schemes can be approximated by discrete-time incentive schemes (see Remark 1). The supremum over discrete-time schemes is therefore at least as large as the supremum over continuous-time schemes. The second inequality, $V \geq V_{CP}$ follows from inclusion because compound Poisson incentive schemes can be approximated from discrete time (the second part of Lemma 19). The third inequality, $V_{CP} \geq V_{DL}$ follows because the Δt-optimal discrete-time incentive scheme corresponds to a compound Poisson incentive scheme with frequency $\lambda := r\frac{e^{-r\Delta t}}{1-e^{-r\Delta t}}$. The compound Poisson incentive scheme attains strictly higher value as shown in the first part of Lemma 19. ■

A.2.3 Properties of value function

Lemma 21 *V is strictly concave and continuous on $[0, u-k)$.*

Proof. Lemma 1 states that $V = V_{DL}$, which is continuous and concave by Corollary 1. I show strict concavity by constructing a compound Poisson incentive scheme that gives value strictly above the convex combination.

For all $W_1, W_2 \in [0, u-k)$ with $W_1 > W_2$ and $\alpha \in (0,1)$, I construct a compound Poisson incentive scheme parametrized by $\lambda, \epsilon > 0$ for $W := \alpha W_1 + (1-\alpha)W_2$. For $i = 1, 2$, take compound Poisson incentive scheme $\mathcal{M}_i$ such that $V(\mathcal{M}_i) > V(W_i) - \epsilon$. The incentive scheme uses a binary compound Poisson monitoring with frequency $\lambda > 0$. Upon arrival, signal L_i is generated with probability p_i and the incentive scheme continues according to $\mathcal{M}_i$. The expected discounted duration before the arrival is $\int_0^\infty re^{-rt}e^{-\lambda t}dt = \frac{r}{r+\lambda}$. The monitoring must satisfy the law of total probability, Bayesian plausibility, promise keeping, and incentive compatibility constraints:

$$\begin{cases} p_1 + p_2 & = 1 \\ p_1(1 - L_1) + p_2(1 - L_2) & = 0 \\ \frac{r}{r+\lambda}(u - k) + \frac{\lambda}{r+\lambda}\left(p_1 W_1 + p_2 W_2\right) & = W \\ \lambda p_1(1 - L_1)(W_1 - W) + \lambda p_2(1 - L_2)(W_2 - W) & = rk\,. \end{cases}$$

For sufficiently high frequency λ, the probabilities are positive and the constraints admit a feasible solution

$$\begin{cases} p_1 & = \frac{(r+\lambda)W - r(u-k) - \lambda W_2}{\lambda(W_1 - W_2)} \\ p_2 & = \frac{(r+\lambda)W - r(u-k) - \lambda W_1}{\lambda(W_2 - W_1)} \\ L_1 & = 1 - \frac{1}{p_1}\frac{rk}{\lambda(W_1 - W_2)} \\ L_2 & = 1 - \frac{1}{p_2}\frac{rk}{\lambda(W_2 - W_1)}\,. \end{cases}$$

In excess of the convex combination $\alpha V(W_1) + (1-\alpha)V(W_2)$, this incentive scheme offers value

$$\begin{aligned} & \frac{r}{r+\lambda}\pi - \frac{\lambda}{r+\lambda}\left(p_1 c(L_1) + p_2 c(L_2)\right) + \frac{\lambda}{r+\lambda}\left(p_1 V(\mathcal{M}_1) + p_2 V(\mathcal{M}_2)\right) \\ & - \left(\alpha V(W_1) + (1-\alpha)V(W_2)\right) \\ > & \frac{r}{r+\lambda}\pi - \frac{\lambda}{r+\lambda}\left(p_1 c(L_1) + p_2 c(L_2)\right) - \frac{\lambda}{r+\lambda}\epsilon \\ & + \frac{W_2 - u + k}{W_1 - W_2}V(W_1) + \frac{W_1 - u + k}{W_2 - W_1}V(W_2) \\ > & \frac{1}{r+\lambda}\left(r\pi - \lambda\left(p_1 c(L_1) + p_2 c(L_2)\right) - \lambda\epsilon - r\left(\pi - c\left(\frac{u-k}{u}\right)\right)\right) \\ = & \frac{1}{r+\lambda}\left(rc\left(\frac{u-k}{u}\right) - \lambda\left(p_1 c(L_1) + p_2 c(L_2)\right) - \lambda\epsilon\right)\,. \end{aligned}$$

In the first inequality, I invoke the ϵ-optimality of $\mathcal{M}_i$. In the second inequality, I use $V(W_2) - V(W_1) \leq \frac{\pi - c\left(\frac{u-k}{u}\right)}{u-k}(W_2 - W_1)$ and $V(W_1) \leq \frac{\pi - c\left(\frac{u-k}{u}\right)}{u-k}W_1$ implied by Lemma 3 and the concavity of V, and then simplify the expression.

Now take $\lambda \to \infty$ and $\epsilon = o(\lambda^{-1}) \to 0$. I note that the monitoring cost

$\lambda\left(p_1 c(L_1) + p_2 c(L_2)\right) = \lambda O\left(\lambda^{-2}\right) = O(\lambda^{-1}) \to 0$ because $c'(1) = 0$ and $c''(1) > 0$. The error is $\lambda\epsilon = \lambda o(\lambda^{-1}) = o(1)$ by construction. Therefore, for sufficiently frequent arrival and small error, the constructed incentive scheme offers strictly higher value than the convex combination. ■

Corollary 2 *$V_{\Delta t} \to V$ uniformly on $[0, W]$ for all $W \in (0, u-k)$.*

Proof. The concavity of $V_{\Delta t}$ and continuity of V implies that the convergence is uniform on compact subsets. ■

Lemma 22 *For $\delta > 0$ and any compound Poisson incentive scheme with initial continuation W_0, define the exit time $\rho := \inf\{t : W_t \notin (W_0 - \delta, W_0 + \delta)\}$. There exist $\theta, \eta > 0$ such that any incentive scheme with $\mathbb{E}\left[1 - e^{-r\rho}\right] < \theta$ attains value at most $V(W_0) - \eta/2$.*

Proof. Because V is strictly concave (Lemma 21), there exists $\eta > 0$ such that $V(W_0) > \left(V(W_0 - \delta) + V(W_0 + \delta)\right)/2 + \eta$. Let $Z_t := u - k - (u - k - W_0)e^{rt}$ be the expected continuation utility at time t. By continuity, there exists $T > 0$ such that for all $t \in [0, T]$

$$V(W_0) - \eta > \frac{(W_0 + \delta) - Z_t}{2\delta} V(W_0 - \delta) + \frac{Z_t - (W_0 - \delta)}{2\delta} V(W_0 + \delta)\,.$$

I first construct random variable Y from $W_{\rho\wedge T}$ by performing a mean-preserving contraction for $W_{\rho\wedge T} \in (W_0 - \delta, W_0 + \delta)$ such that $\mathbb{P}\left[Y \in (W_0 - \delta, W_0 + \delta)\right] = 0$. The Markov inequality implies $\mathbb{P}\left[W_{\rho\wedge T} \in (W_0 - \delta, W_0 + \delta)\right] = \mathbb{P}\left[\rho > T\right] \leq \theta/(1 - e^{-rT})$. The expectations satisfy

$$\mathbb{E}\left[V(W_{\rho\wedge T})\right] \leq \mathbb{E}\left[V(Y)\right] + \frac{2\max|V|}{1 - e^{-rT}}\theta \leq V(W_0) - \eta + \frac{2\max|V|}{1 - e^{-rT}}\theta$$

where the second inequality follows because V is concave and Y has mean $\mathbb{E}[Y] \in [Z_0, Z_T]$ with its support mutually exclusive from $(W_0 - \delta, W_0 + \delta)$.

The expected discounted continuation value thus satisfies

$$\begin{aligned}\mathbb{E}\left[e^{-r\rho\wedge T}V\left(W_{\rho\wedge T}\right)\right] \leq & \mathbb{E}\left[V(W_{\tau\wedge T})\right] + \max|V|\mathbb{E}\left[1-e^{-r\rho}\right] \\ \leq & V(W_0) - \eta + \theta\max|V|\left(1+\frac{2}{1-e^{-rT}}\right).\end{aligned}$$

Take $\theta := \frac{\eta/2}{\pi+\max|V|\left(1+\frac{2}{1-e^{-rT}}\right)}$. The value of the compound Poisson incentive scheme is then bounded from above by

$$\begin{aligned}&\mathbb{E}\left[\int_0^{\rho\wedge T} re^{-rt}\left(\pi - c(L_t)\right)dt + e^{-r\rho\wedge T}V(W_{\tau\wedge T})\right] \\ \leq & \pi\mathbb{E}\left[1-e^{-r\rho}\right] + V(W_0) - \eta + \theta\max|V|\left(1+\frac{2}{1-e^{-rT}}\right) \\ \leq & V(W_0) - \eta + \theta\left(\pi + \max|V|\left(1+\frac{2}{1-e^{-rT}}\right)\right) \\ \leq & V(W_0) - \eta/2\,.\end{aligned}$$

■

A.2.4 Compound Poisson HJB

Definition 2 (Viscosity solution to compound Poisson HJB) *A concave continuous function V is a viscosity solution to the compound Poisson HJB*

$$v(W) = \pi + \sup_{\substack{\lambda,p_i,L_i,J_i\\|i|\leq 4}} \lambda\sum_i p_i\left(v(J_i) - v(W) - c(L_i)\right)$$

$$s.t.\begin{cases}\sum_i p_i & = 0\\ \sum_i p_i(1-L_i) & = 0\\ \lambda\sum_i p_i(J_i - W) & = W - u + k\\ \lambda\sum_i p_i(1-L_i)(J_i-W) & = k\end{cases}$$

if and only if

1. *it is a viscosity subsolution, i.e. any $\phi \in \mathcal{C}^2$ with $\phi \geq V$ and $\phi(W) = V(W)$*

satisfies

$$V(W) \leq \pi + \sup \lambda \sum_i p_i \left(\phi(J_i) - V(W) - c(L_i)\right)$$

2. *it is a viscosity supersolution, i.e. any* $\phi \in \mathcal{C}^2$ *with* $\phi \leq V$ *and* $\phi(W) = V(W)$ *satisfies*

$$V(W) \geq \pi + \sup \lambda \sum_i p_i \left(\phi(J_i) - V(W) - c(L_i)\right)$$

This definition of viscosity solution specializes that of general jump processes in Soner (1988) to the compound Poisson processes.

Proposition 2 *The value function* V *is a viscosity solution to the compound Poisson HJB.*

Proof. I first show that V is a subsolution. It suffices to consider ϕ being concave because the Hamiltonian is the same for the concave envelope of ϕ.[15]

Suppose

$$\phi(W) > \pi + \sup \lambda \sum_i p_i \left(\phi(J_i) - \phi(W) - c(L_i)\right)$$

at $W = W_0 \in (0, u - k)$. I show that the strict inequality holds in a neighborhood of W_0.

Lemma 23 *There exists* $\delta > 0$ *such that*

$$\phi(W) > \pi + \sup \lambda \sum_i p_i \left(\phi(J_i) - \phi(W) - c(L_i)\right)$$

for $W \in [W_0 - \delta, W_0 + \delta]$.

[15] If $\phi(W)$ is not on the concave envelope, then the RHS is infinity and so the inequality is trivially satisfied.

Proof. Suppose otherwise, i.e., there exists a sequence $W_n \to W_0$ such that

$$\phi(W_n) \leq \pi + \sup \lambda \sum_i p_i \left(\phi(J_i) - \phi(W_n) - c(L_i)\right) .$$

I bound the RHS by

$$\begin{aligned}
&\pi + \sup \lambda \sum_i p_i \left(\phi(J_i) - \phi(W_n) - c(L_i)\right) \\
=&\pi + (W_n - u + k)\phi'(W_n) + \sup \lambda \sum_i p_i \left(\phi(J_i) - \phi(W_n) - (J_i - W_n)\phi'(W_n) - c(L_i)\right) \\
\leq&\pi + (W_n - u + k)\phi'(W_n) + \sup \frac{1}{1-L} \frac{k}{J - W_n} \left(\phi(J) - \phi(W_n) - (J - W_n)\phi'(W_n) - c(L)\right) \\
\leq&\pi + (W_n - u + k)\phi'(W_n) + \frac{1}{1-L_n} \frac{k}{J_n - W_n} \left(\phi(J_n) - \phi(W_n) - (J_n - W_n)\phi'(W_n) - c(L_n)\right) \\
&+ \epsilon_n
\end{aligned}$$

where $(L, J) \in \max_i \frac{(1-L_i)(J_i - W)}{-(\phi(J_i) - V(W) - (J_i - W)\phi'(W)) + c(L_i)}$ is a maximizer of the benefit-cost ratio among i's, $\epsilon_n := 2^{-n}$, and (L_n, J_n) is a ϵ_n-maximizer. The equality follows from the promise keeping constraint.

Without loss of generality, I take L_n as the unique maximizer conditional on J_n. It is the solution to the first-order condition

$$c'(L_n)(L_n - 1) - c(L_n) = -\left(\phi(J_n) - \phi(W_n) - (J_n - W_n)\phi'(W_n)\right) .$$

Because $[0, u-k]$ is compact, J_n converges (in a subsequence) on $[0, u-k]$.

First, consider the case where the sequence converges to $J_0 \neq W_0, u-k$. Let L_0 be the conditional maximizer of J_0. Note that $J_0 \neq W_0$ implies $L_0 \neq 1$. Because $\phi, c \in \mathcal{C}^1$ and $\epsilon_n \to 0$, the hypothesis implies the inequality at W_0

$$\phi(W_0) \leq \pi + (W_0 - u + k)\phi'(W_0) + \frac{1}{1-L_0} \frac{k}{J_0 - W_0} \left(\phi(J_0) - \phi(W_0) - (J_0 - W_0)\phi'(W_0) - c(L_0)\right) .$$

I define compound Poisson control $\{\lambda, \{p_i, L_i, J_i\}_{i=1,2}\}$, parametrized by $\lambda > 0$, by

$$
\begin{aligned}
p_1 :=& \frac{1}{\lambda}\frac{1}{1-L_0}\frac{k}{J_0-W_0} \\
p_2 :=& 1-\frac{1}{\lambda}\frac{1}{1-L_0}\frac{k}{J_0-W_0} \\
L_1 :=& 1-(1-L_0)\frac{J_0-W_0}{J_0-W_0-\frac{1}{\lambda}\left(W_0-u+k-\frac{k}{1-L_0}\right)} \\
L_2 :=& 1+\frac{1}{\lambda}\frac{1}{p_2}\frac{k}{J_0-W_0-\frac{1}{\lambda}\left(W_0-u+k-\frac{k}{1-L_0}\right)} \\
J_1 :=& J_0 \\
J_2 :=& W_0+\frac{1}{\lambda}\frac{1}{p_2}\left(W_0-u+k-\frac{k}{1-L_0}\right).
\end{aligned}
$$

It is straightforward to verify that this control satisfies the law of total probability, Bayesian plausibility, promise keeping, and incentive compatibility constraints.

The flow value of this control is

$$
\begin{aligned}
& \pi+\lambda\sum_i p_i\left(\phi(J_i)-\phi(W_0)-c(L_i)\right) \\
=& \pi+\frac{1}{1-L_0}\frac{k}{J_0-W_0}\left(\phi(J_0)-\phi(W_0)-c(L_0)\right) \\
& +\left(\phi'(W_0)(W_0-u+k-\frac{k}{1-L_0})-c'(1)\lambda(L_2-1)\right)+o_\lambda(1) \\
=& \pi+(W_0-u+k)\phi'(W_0)+\frac{1}{1-L_0}\frac{k}{J_0-W_0}\left(\phi(J_0)-\phi(W_0)-(J_0-W_0)\phi'(W_0)-c(L_0)\right)+o_\lambda(1).
\end{aligned}
$$

The first equality follows from $L_1\to L_0$, $\lambda(L_2-1)\to\frac{k}{J_0-W_0}$, and $\lambda\left(J_2-W_0\right)=W_0-u+k-\frac{k}{1-L_0}$. The second equality follows from $c'(1)=0$. I take $\lambda\to\infty$ to obtain

$$
\begin{aligned}
& \pi+\sup\lambda\sum_i p_i\left(\phi(J_i)-\phi(W)-c(L_i)\right) \\
\geq& \pi+(W_0-u+k)\phi'(W_0)+\frac{1}{1-L_0}\frac{k}{J_0-W_0}\left(\phi(J_0)-\phi(W_0)-(J_0-W_0)\phi'(W_0)-c(L_0)\right) \\
\geq& \phi(W_0)
\end{aligned}
$$

which contradicts the subsolution inequality.

Second, consider the case where the sequence converges to $u-k$. Take $J_0 \uparrow u-k$. I define compound Poisson control $\{\lambda, \{p_i, L_i, J_i\}_{i=1,2}\}$, parametrized by J_0 and $\lambda > 0$, as in the first case. Because $\phi, c \in \mathcal{C}^1$, the convergence in flow value is uniform in a neighborhood of $u-k$. Therefore, I take $(\lambda, J_0) \to (\infty, u-k)$ to arrive at the same contradiction.

Third and last, consider the case where the sequence converges to W_0. The first-order condition gives

$$\begin{aligned} c'(L_n)(L_n-1) - c(L_n) =& -\phi''(W_n)(J_n - W_n)^2 + o(|J_n - W_n|^2) \\ 1 - L_n =& \left(\frac{c''(1)}{-\phi''(W_n)}\right)^{\frac{1}{2}} (J_n - W_n) + o(|J_n - W_n|) \,. \end{aligned}$$

Substituting L_n into the flow value gives

$$\begin{aligned} & \pi + (W_n - u + k)\phi'(W_n) + \frac{1}{1-L_n}\frac{k}{J_n - W_n}\left(\phi(J_n) - \phi(W_n) - (J_n - W_n)\phi'(W_n) - c(L_n)\right) + \epsilon_n \\ =& \pi + (W_n - u + k)\phi'(W_n) + k\left((-\phi''(W_n))\, c''(1)\right)^{\frac{1}{2}} + o_{|J_n - W_n|}(1) + \epsilon_n \,. \end{aligned}$$

Because $\phi, c \in \mathcal{C}^2$ and $\epsilon_n \to 0$, I take $n \to \infty$ (and thus $J_n - W_n \to W_0 - W_0 = 0$) to obtain the inequality at W_0

$$\phi(W_0) \le \pi + (W_0 - u + k)\phi'(W_0) + k\left((-\phi''(W_0))\, c''(1)\right)^{\frac{1}{2}} \,.$$

I define compound Poisson control $\{\lambda, \{p_i, L_i, J_i\}_{i=1,2}\}$, parametrized by L_1, by

$$\begin{aligned}
\lambda :=& \left(\frac{c''(1)}{-\phi''(W_0)}\right)^{1/2} \frac{W_0 - u + k + \frac{k}{1-L_1}}{1 - L_1} \\
p_1 :=& \frac{1}{2} \\
p_2 :=& \frac{1}{2} \\
L_2 :=& 2 - L_1 \\
J_1 :=& W_0 + \left(\frac{c''(1)}{-\phi''(W_0)}\right)^{1/2} (1 - L_1) \\
J_2 :=& W_0 - (J_1 - W_0)\left(1 - \frac{2(W_0 - u + k)}{W_0 - u + k + \frac{k}{1-L_1}}\right).
\end{aligned}$$

It is straightforward to verify that this control satisfies the law of total probability, Bayesian plausibility, promise keeping, and incentive compatibility constraints.

The flow value of this control is

$$\begin{aligned}
&\pi + \lambda \sum_i p_i \left(\phi(J_i) - \phi(W_0) - c(L_i)\right) \\
=&\pi + \frac{1}{2}\lambda\left(\phi(J_1) + \phi(W_0 - (J_1 - W_0)) - 2\phi(W_0) + \phi(J_2) - \phi(W_0 - (J_1 - W_0)) - c(L_1) - c(L_2)\right) \\
=&\pi + \frac{1}{2}\lambda\left(\phi''(W_0)(J_1 - W_0)^2 + \phi'(W_0)\left(J_2 - W_0 - (J_1 - W_0)\right) - c''(1)(1 - L_1)^2 + o((1 - L_1)^2)\right) \\
=&\pi + \frac{1}{2}k\phi''(W_0)\frac{J_1 - W_0}{1 - L_1} + \phi'(W_0)(W_0 - u + k) - \frac{1}{2}kc''(1)\frac{1 - L_1}{J_1 - W_0} + o(1) \\
=&\pi + (W_0 - u + k)\phi'(W_0) + k\left((-\phi''(W_0))\, c''(1)\right)^{\frac{1}{2}} + o(1)\,.
\end{aligned}$$

The second equality follows from $(J_2 - W_0)^2 - (J_1 - W_0)^2 = o((1 - L_1)^2)$. The third equality follows from $\lambda(1 - L_1)(J_1 - W_0) \to k$. I take $L_1 \to 1$ to obtain

$$\begin{aligned}
&\pi + \sup \lambda \sum_i p_i \left(\phi(J_i) - \phi(W) - c(L_i)\right) \\
\geq&\pi + \frac{1}{2}k\phi''(W_0)\frac{J_1 - W_0}{1 - L_1} + \phi'(W_0)(W_0 - u + k) - \frac{1}{2}kc''(1)\frac{1 - L_1}{J_1 - W_0} \\
\geq&\phi(W_0)
\end{aligned}$$

which contradicts the subsolution inequality. ■

Define exit time ρ as in Lemma 22. For any compound Poisson incentive scheme, Itô lemma implies

$$\begin{aligned}
&\mathbb{E}\left[e^{-r\rho}\phi(W_\rho)\right] \\
=&\phi(W_0)+\mathbb{E}\left[\int_0^\rho re^{-rt}\left(-\phi(W_t)\right)dt+\sum_{t\le\rho}\phi(W_t)-\phi(W_{t-})\right] \\
=&\phi(W_0)+\mathbb{E}\left[\int_0^\rho re^{-rt}\left(-\phi(W_t)+\lambda_t\sum_i p_{it}\left(\phi(J_{it})-\phi(W_t)\right)\right)dt\right].
\end{aligned}$$

The value of any compound Poisson incentive scheme with $\mathbb{E}\left[1-e^{-r\rho}\right]\ge\theta$ is thus bounded by

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\rho re^{-rt}\left(\pi-\lambda_t\sum_i p_{it}c(L_{it})\right)dt+e^{-r\rho}\phi(W_\tau)\right] \\
=&\phi(W_0)+\mathbb{E}\left[\int_0^\rho re^{-rt}\left(-\phi(W_t)+\pi+\lambda_t\sum_i p_{it}\left(\phi(J_{it})-\phi(W_t)-c(L_{it})\right)\right)dt\right] \\
<&V(W_0)
\end{aligned}$$

where the strict inequality follows from the strictly negative integrand over a set of strictly positive measure. The strict inequality contradicts V as the value of compound Poisson incentive schemes with $\mathbb{E}\left[1-e^{-r\tau}\right]\ge\theta$ (Lemma 22).

I continue to show that V is a supersolution. Suppose there exist control $\lambda,\{(p_i,L_i,J_i)\}$, and $\epsilon>0$ such that

$$V(W)<\pi+\lambda\sum_i p_i\left(\phi(J_i)-V(W)-c(L)\right)-\epsilon$$

for $W_0\in(0,u-k)$. Take $\frac{r}{2(r+\lambda)}\epsilon$-optimal compound Poisson incentive schemes $\mathcal{M}_i$ at $W=J_i$ for each i. Consider the compound Poisson incentive scheme that uses control $\lambda,\{p_i,L_i,J_i\}$ until the first arrival ρ, then switches to $\mathcal{M}_i$ for arrival i. The value of this incentive scheme is

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{\rho} re^{-rt}\left(\pi-\lambda\sum_i p_i c(L_i)\right)dt+e^{-r\rho}\sum_i \mathbf{1}_{W_\rho=J_i}V(\mathcal{M}_i)\right]\\
\geq&\mathbb{E}\left[\int_0^{\rho} re^{-rt}\left(\pi-\lambda\sum_i p_i c(L_i)\right)dt+e^{-r\rho}\sum_i \mathbf{1}_{W_\rho=J_i}V(J_i)\right]-\frac{r}{2\left(r+\lambda\right)}\epsilon\\
\geq&\mathbb{E}\left[\int_0^{\tau} re^{-rt}\left(\pi-\lambda\sum_i p_i c(L_i)\right)dt+e^{-r\rho}\sum_i \mathbf{1}_{W_\rho=J_i}\phi(J_i)\right]-\frac{r}{2\left(r+\lambda\right)}\epsilon\\
=&\phi(W_0)+\mathbb{E}\left[\int_0^{\rho} re^{-rt}\left(-\phi(W)+\pi-\lambda\sum_i p_i\left(\phi(J_i)-\phi(W_0)-c(L_i)\right)\right)dt\right]-\frac{r}{2\left(r+\lambda\right)}\epsilon\\
\geq&\phi(W_0)+\frac{r}{r+\lambda}\epsilon-\frac{r}{2\left(r+\lambda\right)}\epsilon\\
=&V(W_0)+\frac{r}{2\left(r+\lambda\right)}\epsilon
\end{aligned}
$$

where the first equality follows from Itô lemma. This incentive schemes V as the value of compound Poisson incentive schemes. ■

A.2.5 Poisson HJB

Definition 3 (Viscosity solution to Poisson HJB) *A concave continuous function V is a viscosity solution to the (simple) Poisson HJB*

$$
\begin{aligned}
v(W)=&\pi+(W-u+k)v'(W)\\
&+\sup_{L,J}\frac{1}{1-L}\frac{k}{J-W}\left(v(J)-v(W)-(J-W)v'(W)-c(L)\right)\\
&\quad s.t.\ (1-L)(J-W)>0
\end{aligned}
$$

if and only if

1. *it is a viscosity subsolution, i.e. any $\phi\in\mathcal{C}^2$ with $\phi\geq V$ and $\phi(W)=V(W)$*

satisfies

$$V(W) \leq \pi + (W - u + k)\phi'(W) + \sup \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J-W)\phi'(W) - c(L)\right)$$

2. *it is a viscosity supersolution, i.e. any $\phi \in \mathcal{C}^2$ with $\phi \leq V$ and $\phi(W) = V(W)$ satisfies*

$$V(W) \geq \pi + (W - u + k)\phi'(W) + \sup \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J-W)\phi'(W) - c(L)\right)$$

Proof of Proposition 1. I first show that V is a viscosity subsolution to the Poisson HJB. Take $\phi \in \mathcal{C}^2$ with $\phi \geq V$ and $\phi(W) = V(W)$. For any $\{\lambda_i, L_i, J_i\}$, take the maximizer of the benefit-cost ratio $(L, J) \in \max_i \frac{(1-L_i)(J_i-W)}{-(\phi(J_i)-V(W)-(J_i-W)\phi'(W))+c(L_i)}$. The Hamiltonian then satisfies

$$\begin{aligned}
&\pi + \lambda \sum_i p_i \left(\phi(J_i) - V(W) - c(L_i)\right) \\
=&\pi + (W-u+k)\phi'(W) + \lambda \sum_i p_i \left(\phi(J_i) - V(W) - (J_i - W)\phi'(W) - c(L_i)\right) \\
\leq&\pi + (W-u+k)\phi'(W) + \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J-W)\phi'(W) - c(L)\right).
\end{aligned}$$

The equality follows from the promise keeping constraint. The inequality follows from the maximizer (L, J) and the incentive compatibility constraint. Because (L, J) is a feasible control (for the Poisson HJB), I conclude that V inherits the subsolution inequality for the Poisson HJB from the compound Poisson HJB by taking the supremum.

I continue to show that V is a viscosity supersolution to the Poisson HJB by contraposition. Suppose there exist $\phi \leq V$ with $\phi(W) = V(W)$, feasible control

(L, J), and $\epsilon > 0$ such that

$$
\begin{aligned}
V(W) <& \pi + (W - u + k)\phi'(W) \\
&+ \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J-W)\phi'(W) - c(L)\right) - \epsilon\,.
\end{aligned}
$$

Because the Hamiltonian is continuous in L, I have $W - u + k - \frac{k}{1-L} \neq 0$ without loss of generality. For $\alpha > 0$, construct the parametrized (compound) control $\lambda, \{p_i, L_i, J_i\}_{i=1,2}$ defined by $L_1 =: L$, $J_1 := J$, and

$$
\begin{aligned}
\lambda :=& \frac{\alpha}{1-L}\frac{k}{J-W} + \frac{\alpha}{1-\alpha}\frac{W-u+k-\frac{\alpha k}{1-L}}{-(J-W)} \\
p_1 :=& \frac{1}{\lambda}\frac{\alpha}{1-L}\frac{k}{J-W} \\
p_2 :=& \frac{1}{\lambda}\frac{\alpha}{1-\alpha}\frac{W-u+k-\frac{\alpha k}{1-L}}{-(J-W)} \\
L_2 :=& 1 + \frac{\lambda_1}{\lambda_2}(1-L_1) \\
J_2 :=& W + \frac{W-u+k}{\lambda_2} - \frac{\lambda_1}{\lambda_2}(J-W)\,.
\end{aligned}
$$

For $\frac{W-u+k-\frac{\alpha k}{1-L}}{-(J-W)} > 0$, it can verified that the control is feasible for $\alpha > 1$ and sufficiently close to 1. Moreover, it satisfies $\lambda \uparrow \infty$, $p_2 \uparrow 1$, $\lambda p_1 \to \frac{1}{1-L}\frac{k}{J-W}$, $L_2 \to 1$, and $J_2 \to W$ as $\alpha \downarrow 1$. Similarly, for $\frac{W-u+k-\frac{\alpha k}{1-L}}{-(J-W)} < 0$, the control is feasible for $\alpha < 1$ and sufficiently close to 1 with the same asymptotics.

As $\alpha \to 1$ and $\lambda_2 \to \infty$, I have

$$
\phi(J_2) - V(W) - (J_2 - W)\phi'(W) = (J_2 - W)\phi'(W) + o(\lambda_2^{-1}) - (J_2 - W)\phi'(W) = o(\lambda_2^{-1})
$$

and $c(L_2) = o(\lambda_2^{-1})$. Therefore, the Hamiltonian of the compound control satisfies

$$
\begin{aligned}
&\pi + \sum_i \lambda_i \left(\phi(J_i) - V(W) - c(L_i)\right) \\
=&\pi + (W - u + k)\phi'(W) + \sum_i \lambda_i \left(\phi(J_i) - V(W) - (J_i - W)\phi'(W) - c(L_i)\right) \\
=&\pi + (W - u + k)\phi'(W) + \frac{\alpha}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J - W)\phi'(W) - c(L)\right) + o(\lambda_2^{-1}) \\
>&V(W) + \epsilon + o(\lambda_2^{-1})
\end{aligned}
$$

where the first equality follows from the promise keeping constraint. Therefore, V fails the supersolution inequality for the compound Poisson HJB for α sufficiently close to 1, a contradiction. ■

A.2.6 Smoothness of value function

Lemma 24 *The value function is continuously differentiable on* $[0, u - k)$.

Proof. At $W = 0$, $V'(0)$ exists because V is continuous and concave. The derivative at 0 is finite because the first-best incentive scheme has finite derivative.

For $W_0 \in (0, u - k)$, it suffices to show there is no concave kinks because V is concave. Suppose $V'(W_-) > V'(W_+)$. Take $q \in (V'(W_+), V'(W_-))$.

Consider the Hamiltonian

$$
\pi + (W_0 - u + k)q + \sup_{L,J} \frac{1}{1-L}\frac{k}{J-W_0}\left(V(J) - V(W_0) - (J - W_0)q - c(L)\right) .
$$

Because V is right-differentiable at W_0, the value function satisfies

$$
V(J) = V(W_0) + V'(W_+)(J - W_0) + o(J - W_0) .
$$

The FOC for L gives

$$\begin{aligned} c'(L)(L-1) - c(L) &= (q - V'(W_+))(J - W_0) \\ \implies 1 - L &= \left(\frac{q - V'(W_+)}{c''(L)}\right)^{1/2} (J - W_0)^{1/2} + o\left((J - W_0)^{1/2}\right) . \end{aligned}$$

The supremum term therefore satisfies

$$\begin{aligned} &\frac{1}{1-L}\frac{k}{J-W_0}\left(V(J) - V(W_0) - (J - W_0)q - c(L)\right) \\ =&\left(\frac{c''(L)}{q - V'(W_+)}\right)^{1/2}\frac{k}{(J-W_0)^{3/2}}\left(-\left(q - V'(W_0)\right)(J - W_0) - \frac{1}{2}c''(1)\frac{q - V'(W_+)}{c''(1)}(J - W_0)\right) \\ &+ o\left((J - W_0)^{-1/2}\right) \\ =& -\frac{1}{2}kc''(L)^{1/2}\left(q - V'(W_+)\right)^{1/2}(J - W_0)^{-1/2} + o\left((J - W_0)^{-1/2}\right) \end{aligned}$$

and diverges to $-\infty$ as $J \downarrow W_0$. Similarly, the supremum term diverges to $-\infty$ as $J \uparrow W_0$. Therefore, there exists $\delta > 0$ such that the supremum can be taken over $J \in [0, W_0 - \delta] \bigcup [W_0 + \delta, u - k]$. It is attained because the choice of J is compact and the conditional maximizer L is continuous over J.

The kink implies there exists $\phi \geq V$ with $\phi(W_0) = V(W_0)$ such that $\phi'(W_0) = q$. Lemma 2.2 in Soner (1988) implies

$$\begin{aligned} V(W_0) \leq& \pi + (W_0 - u + k)q \\ &+ \max\left\{ \sup_{L, J \in (W_0 - \delta, W_0 + \delta)} \frac{1}{1-L}\frac{k}{J-W_0}\left(\phi(J) - \phi(W_0) - (J - W_0)q - c(L)\right) \right. \\ &\left. \sup_{L, J \in [0, W_0 - \delta] \bigcup [W_0 + \delta, u - k]} \frac{1}{1-L}\frac{k}{J-W_0}\left(V(J) - V(W_0) - (J - W_0)q - c(L)\right) \right\} \\ =& \pi + (W_0 - u + k)q \\ &+ \max_{L, J \in [0, W_0 - \delta] \bigcup [W_0 + \delta, u - k]} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J - W)q - c(L)\right) \end{aligned}$$

Because $[0, W_0 - \delta] \bigcup [W_0 + \delta, u - k]$ is compact and the conditional maximizer L is continuous over J, the supremum is attained.

Because V is concave, there exists a sequence $W \downarrow W_0$ such that $V'(W)$ exists and converges to $V'(W_+)$. The differentiability implies the existence of $\phi \leq V$ with $\phi(W) = V(W)$.

$$
\begin{aligned}
V(W) \geq & \pi + (W - u + k)V'(W) \\
& + \max\left\{ \sup_{L,J\in(W-\delta,W+\delta)} \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - \phi(W) - (J-W)\phi'(W) - c(L)\right) \right. \\
& \left. \sup_{L,J\in[0,W-\delta]\bigcup[W+\delta,u-k]} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J-W)V'(W) - c(L)\right) \right\} \\
\geq & \pi + (W - u + k)V'(W) \\
& + \max_{L,J\in[0,W-\delta]\bigcup[W+\delta,u-k]} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J-W)V'(W) - c(L)\right) .
\end{aligned}
$$

The first inequality follows from Lemma 2.2 in Soner (1988). The second inequality follows from set inclusion. Because $[W + \delta, u - k]$ is compact and the conditional maximizer L is continuous over J, the supremum is attained. Moreover, the strict concavity of V implies the conditional maximizer L is bounded uniformly from away 1. The theorem of maximum thus applies

$$
\begin{aligned}
V(W_0) \geq & \pi + (W_0 - u + k)V'(W_+) \\
& + \max_{L,J\in[0,W-\delta]\bigcup[W+\delta,u-k]} \frac{1}{1-L}\frac{k}{J-W_0}\left(V(J) - V(W_0) - (J-W_0)V'(W_+) - c(L)\right)
\end{aligned}
$$

when $W \downarrow W_0$. The analogous argument from the left of W_0 gives

$$
\begin{aligned}
V(W_0) \geq & \pi + (W_0 - u + k)V'(W_-) \\
& + \max_{L,J\in[0,W-\delta]\bigcup[W+\delta,u-k]} \frac{1}{1-L}\frac{k}{J-W_0}\left(V(J) - V(W_0) - (J-W_0)V'(W_-) - c(L)\right) .
\end{aligned}
$$

I show that the Hamiltonian

$$
\begin{aligned}
H(q) := & \pi + (W_0 - u + k)q \\
& + \max_{L,J\in[0,W-\delta]\bigcup[W+\delta,u-k]} \frac{1}{1-L}\frac{k}{J-W_0}\left(V(J) - V(W_0) - (J-W_0)q - c(L)\right)
\end{aligned}
$$

is strictly convex on $[V'(W_+), V'(W_-)]$. It is weakly convex because max is convex.

Take $q_1 \neq q_2$ and $\alpha \in (0,1)$, and $q_3 := \alpha q_1 + (1-\alpha) q_2$ as the convex combination. Let (L_3, J_3) be the maximizer for q_3, and define the conditional maximizer $\tilde{L}_i$ for $i = 1, 2$ by

$$c'(\tilde{L}_i)(\tilde{L}_i - 1) - c(\tilde{L}_i) = -\left(V(J_3) - V(W_0) - (J_3 - W_0) q_i\right) .$$

Note that $\tilde{L}_i \neq L_3$ because $J_3 \neq W_0$ and $q_i \neq q_3$. The Hamiltonian is strictly convex:

$$\begin{aligned}
& H(q_3) \\
=& \pi + (W_0 - u + k) q_3 + \frac{1}{1 - L_3} \frac{k}{J_3 - W_0} \left(V(J_3) - V(W_0) - (J_3 - W_0) q_3 - c(L_3)\right) \\
=& \alpha \left(\pi + (W_0 - u + k) q_1 + \frac{1}{1 - L_3} \frac{k}{J_3 - W_0} \left(V(J_3) - V(W_0) - (J_3 - W_0) q_1 - c(L_3)\right) \right) \\
& + (1-\alpha) \left(\pi + (W_0 - u + k) q_2 + \frac{1}{1 - L_3} \frac{k}{J_3 - W_0} \left(V(J_3) - V(W_0) - (J_3 - W_0) q_2 - c(L_3)\right) \right) \\
<& \alpha \left(\pi + (W_0 - u + k) q_1 + \frac{1}{1 - \tilde{L}_1} \frac{k}{J_3 - W_0} \left(V(J_3) - V(W_0) - (J_3 - W_0) q_1 - c(\tilde{L}_1)\right) \right) \\
& + (1-\alpha) \left(\pi + (W_0 - u + k) q_1 + \frac{1}{1 - \tilde{L}_2} \frac{k}{J_3 - W_0} \left(V(J_3) - V(W_0) - (J_3 - W_0) q_1 - c(\tilde{L}_2)\right) \right) \\
\leq& \alpha H(q_1) + (1-\alpha) H(q_2) .
\end{aligned}$$

The strict inequality follows from the uniqueness of the conditional maximizer $\tilde{L}_i$. The weak inequality follows from the definition of the Hamiltonian.

From the supersolution inequalities, I have $V(W_0) \geq H(V'(W_+))$ and $V(W_0) \geq H(V'(W_-))$. The strict convexity of H implies $V(W_0) > H(q)$, which contradicts $V(W_0) \leq H(q)$ from the subsolution inequality. ■

Corollary 3 *For $W \in (0, u-k)$ and $J \to W$, the directional derivatives converge $V'_{\Delta t}(J_\pm) \to V'(W)$.*

Proof. I first claim that $V'_{\Delta t}(W_\pm) \to V'(W)$. For $\epsilon > 0$, there exists $h > 0$ such that

$$\frac{V(W+h) - V(W)}{h} > V'(W) - \epsilon .$$

Lemma 1 states that $\lim V_{\Delta t}(W) = V(W)$ so there exists $\delta > 0$ such that $dt < \delta$

implies

$$\frac{V_{\Delta t}(W+h) - V_{\Delta t}(W)}{h} > V'(W) - \epsilon\,.$$

The concavity of $V_{\Delta t}$ implies $V'_{\Delta t}(W_+) \geq \frac{V_{\Delta t}(W+h)-V_{\Delta t}(W)}{h}$. Because ϵ is arbitrary, I have $\liminf V'_{\Delta t}(W_+) \geq V'(W)$. The analogous argument from the left gives $\limsup V'_{\Delta t}(W_-) \leq V'(W)$. The claim then follows from $V'_{\Delta t}(W_+) \leq V'_{\Delta t}(W_-)$.

The continuity of V' implies that, for $\epsilon > 0$, there exists $h > 0$ such that $V'(W + h) > V'(W) - \epsilon$. Because $J \to W$, the concavity of $V_{\Delta t}$ gives $V'_{\Delta t}(J_\pm) \geq V'_{\Delta t}(W + h_-)$ and therefore $\liminf V'_{\Delta t}(J_+) \geq \liminf V'_{\Delta t}(W + h_-) = V'(W + h) > V'(W) - \epsilon$. The analogous argument from the left gives $\limsup V'_{\Delta t}(J_-) < V'(W) + \epsilon$. The result then follows because ϵ is arbitrary. ■

A.3 Optimality of termination upon Poisson arrival

In this subsection, I show the optimality of immediate termination in discrete time and then the same result for Poisson monitoring by continuity. The key results are Lemma 2 and Proposition 3. Lemma 2 shows that the optimal discrete-time incentive scheme must consist of a signal that leads to immediate termination. Proposition 3 establishes the optimality of immediate termination upon Poisson arrival by continuity.

A.3.1 Proof of Lemma 2

For $W_0 \in (0, u - k)$, work is optimal $V_{\Delta t}(W_0) = AV_{\Delta t}(W_0)$ for $\Delta t > 0$ sufficiently small by Lemma 11 and Lemma 14. Take such Δt. At W_0, the maximization (6) admits a maximizer $\{(p_i, L_i, J_i) : 1 \leq i \leq 4\}$ by Lemma 10. Without loss of generality, $p_i > 0$ for all i. All four continuation values $\{J_i\}$ are extreme points of $V_{\Delta t}$ by Lemma 12. Suppose none of the four points is termination, i.e. $J_i \neq 0$ for all i. At each J_i, work is optimal $V_{\Delta t}(J_i) = AV_{\Delta t}(J_i)$ again by Lemma 11. At each J_i, the maximization (6) admits a maximizer $\{(p_{ij}, L_{ij}, J_{ij}) : 1 \leq j \leq 4\}$ again by Lemma 10. Without loss of generality, $p_{ij} > 0$ for all i.

The Δt-value function at W_0 is thus

$$
\begin{aligned}
&V_{\Delta t}(W_0) \\
=&(1-e^{-r\Delta t})(u-k) - \sum_i p_i c(L_i) + e^{-r\Delta t} \sum_i p_i V_{\Delta t}(J_i) \\
=&(1-e^{-r\Delta t})(u-k) - \sum_i p_i c(L_i) \\
&+ e^{-r\Delta t} \sum_i p_i \left((1-e^{-r\Delta t})(u-k) - \sum_j p_{ij} c(L_{ij}) + e^{-r\Delta t} \sum_j p_{ij} V_{\Delta t}(J_{ij}) \right) \\
=&(1+e^{-r\Delta t})(1-e^{-r\Delta t})(u-k) - \sum_i p_i c(L_i) - e^{-r\Delta t} \sum_i p_i \sum_j p_{ij} c(L_{ij}) \\
&+ e^{-2r\Delta t} \sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij}) \,.
\end{aligned}
$$

The second equality follows from the maximizers at J_i's.

For $p_0 := \frac{1}{1+e^{-r\Delta t}}$, consider the control $\{(p_0, 1, W_0), ((1-p_0)p_i p_{ij}, L_i L_{ij}, J_{ij}) : i, 1 \leq j \leq 4\}$ at W_0. I verify the three constraints. It satisfies the Bayesian plausibility constraint

$$
\begin{aligned}
&p_0 \times (1-1) + (1-p_0) \sum_i p_i \sum_j p_{ij}(1 - L_i L_{ij}) \\
=&(1-p_0) \sum_i p_i \sum_j p_{ij}(1 - L_i + L_i - L_i L_{ij}) \\
=&(1-p_0) \sum_i p_i (1-L_i) + (1-p_0) \sum_i p_i L_i \sum_j (1-L_{ij}) \\
=&0 \,.
\end{aligned}
$$

The final equality follows from the Bayesian plausibility of the maximizers at W_0 and J_i's.

The control satisfies the promise keeping constraint

$$
\begin{aligned}
&(1-e^{-r\Delta t})(u-k)+e^{-r\Delta t}\left(p_0W_0+(1-p_0)\sum_i p_i\sum_j p_{ij}J_{ij}\right)\\
=&(1-e^{-r\Delta t})(u-k)+e^{-r\Delta t}p_0W_0+(1-p_0)\sum_i p_i\left(J_i-(1-e^{-r\Delta t})(u-k)\right)\\
=&p_0(1-e^{-r\Delta t})(u-k)+e^{-r\Delta t}p_0W_0+(1-p_0)\frac{1}{e^{-r\Delta t}}\left(W_0-(1-e^{-r\Delta t})(u-k)\right)\\
=&W_0\,.
\end{aligned}
$$

The first equality follows from the promise keeping constraints of the maximizers at J_i's. The second equality follows from the promise keeping constraint of the maximizer at W_0. The third equality follows from the definition of p_0.

The control satisfies the incentive compatibility constraint

$$
\begin{aligned}
&e^{-r\Delta t}(1-p_0)\sum_i p_i\sum_{ij} p_{ij}(1-L_iL_{ij})(W_{ij}-W_0)\\
=&e^{-r\Delta t}(1-p_0)\left(\sum_i p_i(1-L_1)\sum_{ij} p_{ij}(W_{ij}-W_0)+\sum_i p_iL_1\sum_{ij} p_{ij}(1-L_{ij})(W_{ij}-W_0)\right)\\
=&(1-p_0)\left(\sum_i p_i(1-L_i)\left(W_i-(1-e^{-r\Delta t})(u-k)\right)+\sum_i p_iL_i(1-e^{-r\Delta t})k\right)\\
=&(1-p_0)\left(\frac{1-e^{-r\Delta t}}{e^{-r\Delta t}}k+(1-e^{-r\Delta t})k\right)\\
=&(1-e^{-r\Delta t})k\,.
\end{aligned}
$$

The second equality follows from the Bayesian plausibility constraint at W_0, and the promise keeping constraint and the incentive compatibility constraint at J_i's. The third equality follows from the Bayesian plausibility and incentive compatibility constraint at W_0.

Therefore, the constructed control is feasible and the Δt-value function is greater than the value of this control

$$
\begin{aligned}
V_{\Delta t}(W_0) \geq & (1-e^{-r\Delta t})(u-k) - (1-p_0)\sum_i p_i \sum_j p_{ij} c(L_i L_{ij}) \\
& + e^{-r\Delta t}\left(p_0 V_{\Delta t}(W_0) + (1-p_0)\sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij})\right) \\
\geq & (1-e^{-r\Delta t})(u-k) - (1-p_0)\left(\sum_i p_i c(L_i) + \sum_i p_i \sum_j p_{ij} c(L_{ij})\right) \\
& + e^{-r\Delta t}\left(p_0 V_{\Delta t}(W_0) + (1-p_0)\sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij})\right) \\
> & (1-e^{-r\Delta t})(u-k) - (1-p_0)\left(\sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i p_i \sum_j p_{ij} c(L_{ij})\right) \\
& + e^{-r\Delta t}\left(p_0 V_{\Delta t}(W_0) + (1-p_0)\sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij})\right).
\end{aligned}
$$

The second weak inequality follows from compound reduction (Assumption 2). The strict inequality follows from $e^{-r\Delta t} < 1$.

By induction, the Δt-value function satisfies for $N = 1, 2, 3, ...$

$$
\begin{aligned}
V_{\Delta t}(W_0) > & \Bigg[(1-e^{-r\Delta t})(u-k) - (1-p_0)\left(\sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i p_i \sum_j p_{ij} c(L_{ij})\right) \\
& \qquad + e^{-r\Delta t}(1-p_0)\sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij})\Bigg] \sum_{n=0}^{N} \left(e^{-r\Delta t} p_0\right)^n \\
& + \left(e^{-r\Delta t} p_0\right)^{N+1} V_{\Delta t}(W_0)
\end{aligned}
$$

and the difference between the LHS and the RHS is increasing in N. The induction follows by replacing $V_{\Delta t}(W_0)$ on the RHS of induction inequality by the value of the constructed control. Therefore, it follows as $N \to \infty$ that

$$
\begin{aligned}
& V_{\Delta t}(W_0) \\
> & \Bigg[(1-e^{-r\Delta t})(u-k) - (1-p_0)\left(\sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i p_i \sum_j p_{ij} c(L_{ij})\right) \\
& \qquad + e^{-r\Delta t}(1-p_0)\sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij})\Bigg] \sum_{n=0}^{\infty}\left(e^{-r\Delta t}p_0\right)^n \\
= & \Bigg[(1-e^{-r\Delta t})(u-k) - (1-p_0)\left(\sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i p_i \sum_j p_{ij} c(L_{ij})\right) \\
& \qquad + e^{-r\Delta t}(1-p_0)\sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij})\Bigg] \frac{1}{1-e^{-r\Delta t}p_0} \\
= & (1+e^{-r\Delta t})(1-e^{-r\Delta t})(u-k) - \left(\sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i p_i \sum_j p_{ij} c(L_{ij})\right) \\
& + e^{-2r\Delta t}\sum_i p_i \sum_j p_{ij} V_{\Delta t}(J_{ij})
\end{aligned}
$$

where the second equality follows from the definition of p_0. I obtain $V_{\Delta t}(W_0) > V_{\Delta t}(W_0)$, a contradiction.

A.3.2 Value function satisfies Termination HJB

Lemma 25 *Suppose $W \in (0, u-k)$. Let $\{p_i, L_i, J_i\}$ be a maximizer of Equation (6) at W for $\Delta t > 0$. Then $(L, J) \to (1, W)$ in probability as $\Delta t \to 0$, i.e., $\lim_{\Delta t \to 0} \sum_i p_i \mathbf{1}_{|(L_i, J_i) - (1, W)| > \epsilon} = 0$ for all $\epsilon > 0$.*

Proof. For $\epsilon > 0$, the strict concavity (Lemma 21) and differentiability (Lemma 24) of V implies that $|J - W| > \epsilon \implies V(J) - V(W) - V'(W)(J-W) < -\eta$ for some $\eta > 0$. The analogous argument applies to c and thus $|L-1| > \epsilon \implies c(L) > \eta$. The promise keeping constraint implies $\overline{J} := \sum_i p_i J_i = \frac{W-(1-e^{-r\Delta t})(u-k)}{e^{-r\Delta t}} \to W$ as $\Delta t \to 0$. Corollary 3 states that the directional derivative satisfies $V'_{\Delta t}(\overline{J}) \to V'(W)$. Because $V_{\Delta t} \to V$ uniformly (Corollary 2), I have $|J - W| > \epsilon \implies V_{\Delta t}(J) - V_{\Delta t}(W) - V'_{\Delta t}(W)(J-W) > \eta$ for sufficiently small Δt.

The maximizer attains the discrete-time value function

$$\begin{aligned} V_{\Delta t}(W) =& (1-e^{-r\Delta t})\pi - \sum_i p_i c(L_i) + e^{-r\Delta t}\sum_i p_i V_{\Delta t}(J_i) \\ (1-e^{-r\Delta t})V_{\Delta t}(W) =& (1-e^{-r\Delta t})\pi + e^{-r\Delta t}\left(V_{\Delta t}(\overline{J}) - V_{\Delta t}(W)\right) - \sum_i p_i c(L_i) \\ &+ e^{-r\Delta t}\sum_i p_i \left(V_{\Delta t}(J_i) - V_{\Delta t}(\overline{J}) - V'_{\Delta t}(\overline{J})(J_i - \overline{J})\right) \\ \le& (1-e^{-r\Delta t})\pi + e^{-r\Delta t}\left(V_{\Delta t}(\overline{J}) - V_{\Delta t}(W)\right) - \eta \sum_i p_i \mathbf{1}_{|L_i-1|>\epsilon} \\ &- \eta e^{-r\Delta t}\sum_i p_i \mathbf{1}_{|J_i-W|>\epsilon}\,. \end{aligned}$$

The second equality follows from the definition of $\overline{J}$. The inequality follows from the convexity of c and the concavity of $V_{\Delta t}$. By the uniform convergence of $V_{\Delta t} \to V$, I take $\Delta t \to 0$ to obtain

$$0 \le -\eta \limsup_{\Delta t\to 0} \mathbb{P}\left[|L-1| > \epsilon\right] - \eta \limsup_{\Delta t \to 0} \mathbb{P}\left[|J-W| > \epsilon\right]$$

and therefore $\lim_{\Delta t\to 0}\sum_i p_i \mathbf{1}_{|L_i-1|>\epsilon} = \lim_{\Delta t\to 0}\sum_i p_i \mathbf{1}_{|J_i-W|>\epsilon} = 0$. ■

Corollary 4 *There exists i such that (p_i, L_i, J_i) converges (in a subsequence) to $(P, W_0, 1)$ with probability $P \ge \frac{1}{4}$ as $\Delta t \to 0$.*

Lemma 26 *For $W \in (0, u-k)$, let L_0 denote the solution to the first-order condition (3) for termination $J = 0$. Then*

$$(L_0, 0) \in \arg\max_{L,J} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J-W)V'(W) - c(L)\right)$$

subject to $(1-L)(J-W) > 0$.

Proof. Take $\Delta t \to 0$ and let $\{p_i, L_i, J_i\}$ denote maximizer of Equation (6) at W for each Δt. I omit the dependence on Δt for east of notation. I enumerate the four points such that $(p_1, L_1, J_1) \to (P, 1, W)$ for some $P \ge \frac{1}{4}$ as $\Delta t \to 0$ by Corollary 4, and $p_2 > 0$ and $J_2 = 0$ by Lemma 2.

Take L^* and J^* such that $(1-L^*)(J^*-W) > 0$. For each Δt, I construct a control $\{(\tilde{p}_i, \tilde{L}_i, \tilde{J}_i) : 1 \leq i \leq 5\}$ parametrized by $\tilde{p}_5 \geq 0$. It coincides with the maximizer $(\tilde{p}_i, \tilde{L}_i, \tilde{J}_i) = (p_i, L_i, J_i)$ for $i = 3, 4$. It fixes the likelihood ratio and continuation value at $(\tilde{L}_2, \tilde{J}_2) = (L_2, J_2)$ and $(\tilde{L}_5, \tilde{J}_5) = (L^*, J^*)$ for $i = 2, 5$. Because the maximizer satisfies the four constraints—the law of total probability, Bayesian plausibility, promise keeping, and incentive compatibility constraints—the constructed control is required to satisfy

$$\sum_i \tilde{p}_i = \sum_i p_i$$
$$\sum_i \tilde{p}_i \left(1 - \tilde{L}_i\right) = \sum_i p_i \left(1 - L_i\right)$$
$$\sum_i \tilde{p}_i \tilde{J}_i = \sum_i p_i J_i$$
$$\sum_i \tilde{p}_i \left(1 - \tilde{L}_i\right) \left(\tilde{J}_i - W\right) = \sum_i p_i (1 - L_i)(J_i - W)$$

where the summation is over $1 \leq i \leq 5$ and I write $(p_5, L_5, J_5) := (0, L^*, J^*)$.

Because the maximizer satisfies the continuously differentiable constraints, the implicit function theorem states that the system of constraints admits a solution $\{(\tilde{p}_i, \tilde{L}_i, \tilde{J}_i) : 1 \leq i \leq 5\}$ continuously differentiable with respect to parameter $\tilde{p}_5$ in a neighborhood of 0 with derivatives

$$\frac{d\tilde{p}_2}{d\tilde{p}_5} = -\frac{(L_1 - L^*)(J^* - J_1)}{(L_1 - L_2)(0 - J_1)}$$
$$\frac{d\tilde{p}_1}{d\tilde{p}_5} = \frac{(L_1 - L^*)(J^* - J_1)}{(L_1 - L_2)(0 - J_1)} - 1$$
$$\frac{d\tilde{L}_1}{d\tilde{p}_5} = -\frac{1}{p_1} \frac{J^* - 0}{0 - J_1}(L_1 - L^*)$$
$$\frac{d\tilde{J}_1}{d\tilde{p}_5} = \frac{1}{p_1} \frac{L_1 - L^*}{L_1 - L_2}(J^* - J_1) - \frac{1}{p_1}(J^* - J_1)$$

at $\tilde{p}_5 = 0$.

The marginal value of this control with respect to $\tilde{p}_5$ is

$$
\begin{aligned}
&\frac{d}{d\tilde{p}_5}\left(-\sum_i \tilde{p}_i c(\tilde{L}_i) + e^{-r\Delta t}\sum_i \tilde{p}_i V_{\Delta t}(\tilde{J}_i)\right)\\
=&-\frac{d\tilde{p}_1}{d\tilde{p}_5}c(L_1) - p_1 c'(L_1)\frac{d\tilde{L}_1}{d\tilde{p}_5} - \frac{d\tilde{p}_2}{d\tilde{p}_5}c(L_2) - c(L^*)\\
&+ e^{-r\Delta t}\left(\frac{d\tilde{p}_1}{d\tilde{p}_5}V_{\Delta t}(J_1) + p_1 V'_{\Delta t}(J_1)\frac{d\tilde{J}_1}{d\tilde{p}_5} + \frac{d\tilde{p}_2}{d\tilde{p}_5}V_{\Delta t}(0) + V_{\Delta t}(J^*)\right)\\
=&(L_1 - L^*)\frac{J^* - J_1}{k}\left(\frac{1}{L_1 - L^*}\frac{k}{J^* - J_1}\left(e^{-r\Delta t}\left(V_{\Delta t}(J^*) - V_{\Delta t}(J_1) - (J^* - J_1)V'_{\Delta t}(J_1)\right) - c(L^*)\right)\right.\\
&\qquad\qquad\left. - \frac{1}{L_1 - L_2}\frac{k}{0 - J_1}\left(e^{-r\Delta t}\left(V_{\Delta t}(0) - V_{\Delta t}(J_1) - (0 - J_1)V'_{\Delta t}(J_1)\right) - c(L_2)\right)\right)\\
&-\frac{d\tilde{p}_1}{d\tilde{p}_5}c(L_1) - p_1 c'(L_1)\frac{d\tilde{L}_1}{d\tilde{p}_5}.
\end{aligned}
$$

I claim that, as $\Delta t \to 0$, the likelihood ratio L_2 converges to L_0. Suppose otherwise, i.e., there exists $\epsilon > 0$ such that $|L_2 - L_0| > \epsilon$ along a subsequence. Because L_0 is the unique maximizer and c is convex, there exists $\eta > 0$ such that

$$
\begin{aligned}
&\frac{1}{1 - L_0}\frac{k}{0 - W}\left(V(0) - V(W) - V'(W)(0 - W) - c(L_0)\right)\\
>&\frac{1}{1 - L}\frac{k}{0 - W}\left(V(0) - V(W) - V'(W)(0 - W) - c(L)\right) + \eta
\end{aligned}
$$

for all L such that $|L - L_2| > \epsilon$. Take $L^* = L_0$ and $J^* = 0$. Recall that $V_{\Delta t} \to V$ uniformly on compact sets by Corollary 2, and $V_{\Delta t}(J_1) \to V'(W)$ by Corollary 3. Because $e^{-r\Delta t} \to 1$, $(L_1, J_1) \to (1, W)$, and $c(1) = c'(1) = 0$, the marginal value of the constructed control with respect to $\tilde{p}_5$ satisfies

$$
\begin{aligned}
&\frac{d}{d\tilde{p}_5}\left(-\sum_i \tilde{p}_i c(\tilde{L}_i) + e^{-r\Delta t}\sum_i \tilde{p}_i V_{\Delta t}(\tilde{J}_i)\right)\\
\geq&(1 - L_0)\frac{0 - W}{k}\frac{\eta}{2}.
\end{aligned}
$$

Therefore, the constructed control attains higher value than the maximizer for sufficiently small $\tilde{p}_5$ and Δt, a contradiction.

Now I consider general (L^*, J^*) such that $(1-L^*)(J^*-W) > 0$. Because $L_2 \to L_0$, the marginal value of the constructed control converges to

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{d}{d\tilde{p}_5} \left(-\sum_i \tilde{p}_i c(\tilde{L}_i) + e^{-r\Delta t} \sum_i \tilde{p}_i V_{\Delta t}(\tilde{J}_i) \right) \\
=&(1-L^*)\frac{J^*-W}{k}\Bigg(\frac{1}{1-L^*}\frac{k}{J^*-W}\left(V(J^*) - V(W) - (J^*-J_1)V'(W) - c(L^*)\right) \\
&\qquad\qquad - \frac{1}{1-L_0}\frac{k}{0-W}\left(V(0) - V(W) - (0-W)V'(W) - c(L_0)\right)\Bigg).
\end{aligned}$$

By the optimality of the maximizer, the limit must be non-positive, which implies the lemma. ■

Proposition 3 (Termination HJB) *The value function V is a classical solution to the termination HJB*

$$\begin{aligned}
v(W) =&\pi + (W-u+k)v'(W) \\
&+ \max_L \frac{1}{1-L}\frac{k}{0-W}\left(v(0) - v(W) - (0-W)v'(W) - c(L)\right) \quad \forall W \in (0, u-k)\,.
\end{aligned}$$

Proof. Because V is concave, it is twice differentiable almost everywhere by the Alexandrov theorem. Consider $W \in (0, u-k)$ such that $V''(W)$ exists.

For $\epsilon > 0$, take $\phi \geq V$ with $\phi(W) = V(W)$ such that $\phi''(W) \in (V''(W), V''(W) + \epsilon)$. Such ϕ exists due to Stone-Weierstrass theorem. It also satisfies $\phi'(W) = V'(W)$ because V is differentiable (Lemma 24). For any $\delta > 0$, Lemma 2.2 in Soner (1988) applied to the subsolution inequality for the Poisson HJB implies

$$\begin{aligned}
V(W) \leq&\pi + (W-u+k)V'(W) \qquad (8)\\
&+ \max\Bigg\{ \sup_{\substack{L,J \\ |J-W|<\delta}} \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J-W)V'(W) - c(L)\right), \\
&\qquad \sup_{\substack{L,J \\ |J-W|\geq\delta}} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J-W)V'(W) - c(L)\right)\Bigg\}.
\end{aligned}$$

Because $\phi \in \mathcal{C}^2$, the cost of information exposure is

$$\phi(J) - V(W) - (J-W)\phi'(W) = \frac{1}{2}\phi''(W)(J-W)^2 + o\left((J-W)^2\right) .$$

Therefore, the first-order condition for L gives

$$\begin{aligned} c'(L)(L-1) - c(L) =& -\frac{1}{2}\phi''(W)(J-W)^2 + o\left((J-W)^2\right) \\ \frac{1}{2}c''(1)(1-L)^2 =& -\frac{1}{2}\phi''(W)(J-W)^2 + o\left((J-W)^2\right) \\ \left(\frac{1-L}{J-W}\right)^2 =& \frac{-\phi''(W)}{c''(1)} + o(1) . \end{aligned}$$

Substituting the FOC into the supremum, I have for $|J-W| < \delta$

$$\begin{aligned} &\sup_{\substack{L,J \\ |J-W|<\delta}} \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J-W)V'(W) - c(L)\right) \\ =&\sup \left(\frac{c''(1)}{-\phi''(W)}\right)^{\frac{1}{2}} \frac{k}{(J-W)^2}\left(\frac{1}{2}\phi''(W)(J-W)^2 - \frac{1}{2}c''(1)\frac{-\phi''(W)}{c''(1)} + o\left((J-W)^2\right)\right) \\ =&k\left(c''(1)\left(-\phi''(W)\right)\right)^{\frac{1}{2}} + o_\delta(1) . \end{aligned}$$

The same operations on V gives

$$k\left(c''(1)\left(-V''(W)\right)\right)^{\frac{1}{2}} = \sup_{\substack{L,J \\ |J-W|<\delta}} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J-W)V'(W) - c(L)\right) + o_\delta(1) .$$

Because $\phi''(W) \in (V''(W), V''(W)+\epsilon)$, I have

$$\begin{aligned} &\sup_{\substack{L,J \\ |J-W|<\delta}} \frac{1}{1-L}\frac{k}{J-W}\left(\phi(J) - V(W) - (J-W)V'(W) - c(L)\right) \\ =&\sup_{\substack{L,J \\ |J-W|<\delta}} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J-W)V'(W) - c(L)\right) + O(\epsilon) + o_\delta(1) . \end{aligned}$$

Therefore, Equation (8) reads

$$\begin{aligned}V(W) \leq &\pi + (W-u+k)V'(W) \\ &+ \sup_{L,J} \frac{1}{1-L}\frac{k}{J-W}\left(V(J) - V(W) - (J-W)V'(W) - c(L)\right) + O(\epsilon) + o_\delta(1) \\ \leq &\pi + (W-u+k)V'(W) \\ &+ \max_{L} \frac{1}{1-L}\frac{k}{0-W}\left(V(0) - V(W) - (0-W)V'(W) - c(L)\right) + O(\epsilon) + o_\delta(1)\end{aligned}$$

where the second inequality follows from Lemma 26. Because ϵ and δ are arbitrary, I obtain

$$\begin{aligned}V(W) \leq &\pi + (W-u+k)V'(W) \\ &+ \max_{L} \frac{1}{1-L}\frac{k}{0-W}\left(V(0) - V(W) - (0-W)V'(W) - c(L)\right) .\end{aligned}$$

Using the supersolution inequality and taking $\phi''(W) \in (V''(W) - \epsilon, V''(W))$, I obtain the reverse inequality similarly. Therefore, the value function V solves the termination HJB whenever it is twice differentiable.

Because $V \in \mathcal{C}^1$ (Lemma 24), it also solves the termination HJB for all $W \in (0, u-k)$ by continuity. ■

A.4 Verification by explicit construction

In this subsection, I construct the candidate value function and incentive scheme, and then verify the scheme's optimality. The argument is standard.

A.4.1 Candidate value function

I construct the candidate value function V^* by solving the termination HJB and FOC (3) with boundary conditions $v(0) = 0$ and $\lim_{W \to u-k} v'(W) = -\infty$.

I first derive an ODE of L as a function of W and study its solutions. By rewriting the termination HJB (Proposition 3) and FOC (3), I obtain the value v and marginal value v' as functions of L.

$$\begin{cases} v(W) & = \frac{1}{u-k}\left(-(0-W)\pi - \left(W-u+k+\frac{k}{L-1}\right)c'(L)(L-1) + (W-u+k)c(L)\right) \\ v'(W) & = \frac{1}{u-k}\left(\pi - \left(1+\frac{1}{1-L}\frac{k}{0-W}\right)c'(L)(L-1) + c(L)\right) \end{cases} \tag{9}$$

Taking derivative of v with respect to W and equating it to v', I obtain a first-order ordinary differential equation of L as a function of W

$$\dot{L} = -\frac{k}{0-W}\frac{c'(L)}{c''(L)}\frac{1}{(L-1)(W-u+k)+k}. \tag{10}$$

Taking derivative of v' with respect to W and substituting $\dot{L}$ with the ODE, I obtain v'' as a function of L

$$v''(W) = -\frac{k}{(0-W)^2}c'(L)\frac{1}{(W-u+k)+\frac{k}{L-1}}. \tag{11}$$

For the candidate value function to be strictly concave, the drift $r\left((W-u+k)+\frac{k}{L-1}\right)$ must be positive. As a result, $\dot{L} > 0$.

Lemma 27 *Consider the initial value problem of ODE (10) at* (W_0, L_0) *on the domain* $W_0 \in (0, u-k]$, $L_0 > 1$, *and* $W - u + k + \frac{k}{L-1} > 0$. *It admits a unique solution on* $(0, W_0]$ *and the solution satisfies* $\lim_{W\to 0} L(W) = 1$.

Proof. The RHS of Equation (10) satisfies the Picard-Lindelöf conditions, i.e. continuous in W and Lipschitz continuous in L on all compact subsets of the domain, so there exists a unique solution in the interior. Moreover, $\dot{L} \in (0, \infty)$ so the solution also solves the initial value problem

$$\dot{W} = \dot{L}^{-1} = -\frac{0-W}{k}\frac{c''(L)}{c'(L)}\left((L-1)(W-u+k)+k\right) \tag{12}$$

with the same initial condition and domain.

Because the solution exists and is monotonic in the interior, it suffices to show the solution approach the boundary of the domain only at $(W = 0, L = 1)$.

First, I consider $(W_1 > 0, L_1 = 1)$. The function $L(W) = 1$ on $W \in [W_1, u-k]$ solves Equation (10) at (W_1, L_1). Because the ODE satisfies the Picard-Lindelöf conditions in the neighborhood of (W_1, L_1), this solution is unique and cannot satisfy the initial condition (W_0, L_0).

Second, I consider $(W_1 = 0, L_1 > 1)$. The function $W(L) = 0$ on $L \in \left[L_1, \frac{u}{u-k}\right]$ solves Equation (12) at (W_1, L_1). Because the ODE satisfies the Picard-Lindelöf conditions in the neighborhood of (W_1, L_1), this solution is unique and cannot satisfy the initial condition (W_0, L_0).

Third, I consider (W_1, L_1) such that $W_1 - u + k + \frac{k}{L_1 - 1} = 0$. Suppose the solution approaches (W_1, L_1). Because Equation (12) satisfies the Picard-Lindelöf conditions in the neighborhood of (W_1, L_1), the unique solution satisfies $\lim_{L \to L_1} \dot{W}(L) = \dot{W}(L_1) = 0$. The curve $W - u + k + \frac{k}{L-1}$ has strictly positive slope at L_1, so the solution cannot approach (W_1, L_1) from above, a contradiction. ■

Corollary 5 *Suppose* (W_0, L_0) *satisfies* $W_0 \in (0, u-k]$*,* $L_0 > 1$*, and* $W - u + k + \frac{k}{L-1} > 0$*. Then the initial value problem of ODE (10) at* (W_0, L_0) *is defined on* $(0, W_0]$*.*

Moreover, suppose L_1 *and* L_2 *are distinct solutions with different initial conditions. Then either* $L_1(W) > L_2(W)$ *or* $L_1(W) < L_2(W)$ *for all* W *such that* L_1 *and* L_2 *are defined.*

Proof. Each solution is defined on $(0, W_0]$ due to Lemma 27. The solutions are ordered because they are continuous on the interval and cannot intersect by the uniqueness of solutions to ODE on the domain. ■

Now, I construct from the initial value problems the likelihood ratio function L^* corresponding to the candidate value function V^*.

For $L_0 \in (0, \infty)$, denote $\underline{L}(\cdot; L_0)$ as the solution to Equation (10) and initial condition $(u-k, L_0)$.

For $W_0 \in (0, u-k)$, define $L_0 := \frac{u-W}{u-k-W}$. It thus holds that $W_0 - u + k + \frac{k}{L_0 - 1} = 0$. Equation (12) satisfies the Picard-Lindelöf conditions in a neighborhood of (W_0, L_0) and thus admits a unique solution. It extends to $L \in (1, L_1)$ by Lemma 27 and $\dot{W} \in (0, \infty)$ in the interior. The solution is strictly monotonic and thus invertible. I denote the inverse as $\overline{L}(\cdot; W_0)$.

For $\epsilon > 0$, consider the compact domain

$$\left\{(W, L) : W \in [\epsilon, u - k - \epsilon], L \in [\underline{L}(W; 2), \overline{L}(W; u - k - \epsilon/2)]\right\} .$$

For all $L_0 > 2$, the function $\underline{L}(W; L_0)$ lies on the domain for all $W \in [\epsilon, u-k-\epsilon]$. It is greater than the lower bound $\underline{L}(W; 2)$ because of $\underline{L}(u-k; L_0) = L_0 > 2 = \underline{L}(u-k; 2)$ and Corollary 5. It is less than the upper bound $\overline{L}(W; u - k - \epsilon/2)$ because of $\overline{L}(u - k - \epsilon; u - k - \epsilon/2) = \frac{k+\epsilon/2}{\epsilon/2} > \underline{L}(u - k - \epsilon/2; L_0)$ and Corollary 5.

For $W \in [\epsilon, u - k - \epsilon]$, the function $\underline{L}(W; L_0)$ is bounded and increasing in L_0 so it admits a limit $L^*(W)$. Because the domain is compact and bounded away from $W = 0$ and $W - u + k + \frac{k}{L-1} = 0$, the derivative $\dot{L}$ is uniformly bounded. Therefore, the functions $\{\underline{L}(\cdot; L_0) : L_0 > 2\}$ are uniformly Lipschitz. The Arzelà-Ascoli theorem implies that the limit L^* is also uniformly Lipschitz. Therefore, L^* is a solution to Equation (10) by the uniqueness of the initial value problem.

The defintion of L^* is consistent for all $\epsilon > 0$ because its definition (for fixed W) is the same. Because it is a solution to Equation (10) on $[\epsilon, u - k - \epsilon]$ for all $\epsilon > 0$, it is a solution on $(0, u - k)$.

From L^*, I construct the candidate value function V^* by Equation (9).

Lemma 28 *The function L^* satisfies $\lim_{W\to 0} L^*(W) = 0$ and $\lim_{W\to u-k} L^*(W) = \infty$. The corresponding V^* defined by Equation (9) is uniformly bounded and satisfies $\lim_{W\to 0} V^*(W) = 0$ and $\lim_{W\to u-k}(V^*)'(W) = -\infty$.*

Proof. L^* satisfies $\lim_{W\to 0} L^*(W) = 0$ by Lemma 27. Moreover, it satisfies $\lim_{W\to u-k} L^*(W) = \infty$ because $L^* \geq \underline{L}(\cdot; L_0)$ which attains L_0 at $u - k$ and is continuous for all $L_0 > 2$.

Rewriting Equation (9), I obtain

$$V(W) = \frac{1}{u-k}\left(\pi W - kc'(L) - (W - u + k)\left(c'(L)(L-1) - c(L)\right)\right)$$

which converges to zero as $W \to 0$ and thus $L \to 1$. Moreover, the function satisfies as $W \to u - k$

$$V(W) \geq \frac{1}{u-k}\left(\pi W - kc'(L)\right) \to \pi - \frac{k}{u-k}c'(\infty) > -\infty\,.$$

When $W \to u - k$, I have $L \to \infty$ and

$$\begin{aligned} V'(W) =& \frac{1}{u-k}\left(\pi - \left(c'(L)(L-1) - c(L)\right) - \frac{k}{W}c'(L)\right) \\ &\to \frac{1}{u-k}\left(\pi - \lim_{L\to\infty}\left(c'(L)(L-1) - c(L)\right) - \frac{k}{u-k}c'(\infty)\right) \\ &= -\infty\,. \end{aligned}$$

Therefore, $\lim_{W\to u-k} V'(W) = -\infty$.

Because V^* is concave and bounded on the boundary, it is uniformly bounded. ■

Lemma 29 *For given $W_0 \in (0, u-k)$, the monitoring cost $\frac{1}{1-L(W)}\frac{k}{0-W}c(L(W))$ is uniformly bounded on $[W_0, u-k)$.*

Proof. Because c is convex with $c'(1) = 0$, the function $c(L)/(L-1)$ is increasing in L. As $L \to \infty$, its limit is

$$\lim_{L\to\infty}\frac{c(L)}{L-1} = \lim_{L\to\infty}\frac{\int_1^L c'(l)dl}{L-1} = \lim_{L\to\infty}\int_0^1 c'(1+(L-1)x)dx = c'(\infty)$$

by the monotone convergence theorem. Therefore, the monitoring cost $\frac{1}{1-L(W)}\frac{k}{0-W}c(L(W))$ is bounded uniformly by $\frac{k}{W_0}c'(\infty)$. ■

Lemma 30 *The ODE $\frac{dW}{dt} = r\left(W - u + k - \frac{k}{1-L^*(W)}\right)$ with initial condition $W_0 \in (0, u-k)$ at $t = 0$ admits a unique solution on $t \geq 0$. Moreover, $W_t \in (0, u-k)$ for all $t \geq 0$.*

Proof. For any $\epsilon > 0$, the ODE admits a unique solution on $t \geq 0$ and $W \in [\epsilon, u-k-\epsilon]$ by the Picard-Lindelöf theorem.

Because $W - u + k - \frac{k}{1-L^*(W)}$ is positive, W is increasing and thus $W_t \geq W_0 > 0$ for all t.

The time τ at which $W_\tau = u - k - \epsilon$ is

$$\begin{aligned}
\int_0^\tau dt =& \int_{L^*(W_0)}^{L^*(u-k-\epsilon)} \frac{dt}{dW}\frac{dW}{dL}dL \\
=& \int_{L^*(W_0)}^{L^*(u-k-\epsilon)} \frac{1}{r\left(W - u + k + \frac{k}{L-1}\right)} \frac{W}{k}\frac{c''(L)}{c'(L)} \left((L-1)(W-u+k)+k\right) dL \\
\geq& \frac{W_0}{rkc'(\infty)} \int_{L^*(W_0)}^{L^*(u-k-\epsilon)} c''(L)dL \\
=& \frac{W_0}{rkc'(\infty)} \Big(\left(c'(L^*(u-k-\epsilon))(L^*(u-k-\epsilon)-1) - c(L^*(u-k-\epsilon))\right) \\
& - \left(c'(L^*(W_0))(L^*(W_0)-1) - c(L^*(W_0))\right) \Big) .
\end{aligned}$$

I have substituted the drift $\frac{dW}{dt}$ and the derivative $\frac{dL}{dW}$ in the second equality, W and $c'(L)$ in the inequality.

Because $\lim_{\epsilon \to 0} L^*(u-k-\epsilon) = \infty$, the Inada condition implies $\tau \to \infty$ as $\epsilon \to 0$, and thus $W_t < u - k$ for all $t \geq 0$. ∎

A.4.2 Verification for Termination HJB

Fix $W_0 \in (0, u-k)$.

Define incentive scheme $\mathcal{M}^* = (\Omega^*, \mathbb{F}^*, \mathbb{P}^*, \Gamma^*, h^*, a^*)$ as follows. Define W_t^* as in Lemma 30, $L_t^* := L^*(W_t^*)$, and $\lambda_t^* := \frac{r}{1-L_t^*}\frac{k}{0-W_t^*}$. Let τ^* denote a random variable with law $\mathbb{P}^*[\tau > t] = e^{-\int_0^t \lambda_s^* ds}$. Define $\Gamma_t^* := \int_0^{t\wedge\tau^*} \lambda_s^*(1-L_s^*)ds + \mathbf{1}_{\tau^* \leq t}(L_{\tau^*}^* - 1)$, $h_t^* = a_t^* := \mathbf{1}_{t \leq \tau^*}$. Let $(\Omega^*, \mathbb{F}^*, \mathbb{P}^*)$ denote the augmented natural filtration of (Γ^*, t).

The cumulative excess likelihood ratio Γ is a càdlàg martingale, and employment h and effort a are left-continuous and thus predictable. I verify that W^* is the agent's continuation value and that the incentive scheme is incentive compatible. For effort

a' and $\tau, T > t$, the Itô lemma gives

$$
\begin{aligned}
& \mathbb{E}_t^{*,a'}\left[e^{-r(T-t)}W_T^*\right] \\
=& W_t^* + \mathbb{E}_t^{*,a'}\left[\int_t^{T\wedge\tau^*} -re^{-r(s-t)}W_s^* ds - \mathbf{1}_{\tau^*\le T}e^{-r(\tau^*-t)}W_{\tau^*-}^*\right] \\
=& W_t^* + \mathbb{E}_t^*\left[\int_t^T -\exp\left(-r(s-t) - \int_t^s \lambda^*(1+(L^*-1)\mathbf{1}_{a'=0})\right)\left(r+\lambda_s^*(1+(L_s^*-1)\mathbf{1}_{a_s'=0})\right)W_s^* ds\right].
\end{aligned}
$$

The second equality follows from the probability law of τ and the change of measure. I add the flow payoff to obtain

$$
\begin{aligned}
& \mathbb{E}_t^{*,a'}\left[\int_t^T e^{-r(s-t)}h_s^*(u-ka_s')ds + e^{-r(T-t)}W_T^*\right] \\
=& W_t^* + \mathbb{E}_t^*\Bigg[\int_t^T \exp\left(-r(s-t) - \int_t^s \lambda^*(1+(L^*-1)\mathbf{1}_{a'=0})\right) \\
& \qquad \left(-\left(r+\lambda_s^*(1+(L_s^*-1)\mathbf{1}_{a_s'=0})\right)W_s^* + r(u-k\mathbf{1}_{a_s'=1})\right)ds\Bigg] \\
=& W_t^* + \mathbb{E}_t^*\Bigg[\int_t^T -r\exp\left(-r(s-t) - \int_t^s \lambda^*(1+(L^*-1)\mathbf{1}_{a'=0})\right) \\
& \qquad \left(W_s^* - u + k - \frac{k}{1-L_s^*}\right)ds\Bigg] \\
=& W_t^*.
\end{aligned}
$$

The first equality follows from the definition of h^* and the change of measure, the second from the definition of λ^*, and the third from that of W^*. Because W is bounded and $W_\tau = 0$ for $\tau < \infty$, the dominated convergence theorem for $T \to \infty$ gives

$$
\mathbb{E}_t^{*,a'}\left[\int_t^\infty e^{-r(s-t)}h_s^*(u-ka_s^*)ds\right] = W_t^*.
$$

Because the agent gets value W_t^* for all effort a', the incentive scheme is incentive compatible and W^* is the agent's continuation value.

Define incentive scheme $\mathcal{M} = (\Omega, \mathbb{F}, \mathbb{P}, \Gamma, h, a)$ as follows. Define $L(W)$ by FOC (3) and W_t by $\frac{d}{dt}W = r\left(W - u + k - \frac{k}{1-L(W)}\right)$ and $W_0 = 0$ as in Lemma 30. I write

$L_t := L(W_t)$ and define $\lambda_t := \frac{r}{1-L_t}\frac{k}{0-W_t}$. Let τ denote a random variable with law $\mathbb{P}[\tau > t] = e^{-\int_0^t \lambda_s ds}$. Define $\Gamma_t := \int_0^{t\wedge\tau} \lambda_s(1-L_s)ds + \mathbf{1}_{\tau\leq t}(L_\tau - 1)$, $h_t \equiv a_t := \mathbf{1}_{t\leq\tau}$. Let $(\Omega, \mathbb{F}, \mathbb{P})$ denote the augmented natural filtration of (Γ, t).

It can be shown, analogous to $\mathcal{M}^*$, that $\mathcal{M}$ is an incentive compatible incentive scheme and W is the agent's continuation value.

I show $\mathcal{M}^*$ attains value $V^*(W_0)$, which is higher than that of $\mathcal{M}$. Because V^* is continuous and concave, I apply Itô formula for semimartingales (Proposition 8.19 in Tankov (2003)) to obtain

$$
\begin{aligned}
&\mathbb{E}^*\left[e^{-rt}V^*(W_t^*)\right] \\
=&V^*(W_0^*) + \mathbb{E}^*\left[\int_0^{t\wedge\tau^*} e^{-rs}\left(-rV^*ds + (V^*)'d(W^*)^c\right) + \mathbf{1}_{\tau^*\leq t}e^{-r\tau^*}(V^*(0) - V^*(W_{\tau^*-}))\right] \\
=&V^*(W_0^*) + \mathbb{E}^*\Bigg[\int_0^t re^{-rs-\int_0^s \lambda^*}\Bigg(- V^* + (W^* - u + k)(V^*)' \\
&\qquad + \frac{1}{1-L^*}\frac{k}{0-W^*}\left(V^*(0) - V^*(W^*) - (0-W^*)(V^*)'\right)\Bigg)ds\Bigg]
\end{aligned}
$$

where $(W^*)^c$ is the continuous part of W^*.

Adding the flow payoffs, I obtain

$$
\begin{aligned}
&\mathbb{E}^*\left[e^{-rt}V^*(W_t^*) + \int_0^t re^{-rs}h^*\left(\pi a^* - \lambda^* c(L^*)\right)ds\right] \\
=&V^*(W_0^*) + \mathbb{E}^*\Bigg[\int_0^t re^{-rs-\int_0^s \lambda^*}\Bigg(- V^* + \pi + (W^* - u + k)(V^*)' \\
&\qquad + \frac{1}{1-L^*}\frac{k}{0-W^*}\left(V^*(0) - V^*(W^*) - (0-W^*)(V^*)' - c(L^*)\right)\Bigg)\Bigg]
\end{aligned}
\tag{13}
$$

Because the integrand is zero by the construction of V^*, I have

$$
\mathbb{E}^*\left[e^{-rt}V^*(W_t^*) + \int_0^t re^{-rs}h^*\left(\pi a^* - \lambda^* c(L^*)\right)ds\right] = V^*(W_0^*) = V^*(W_0)\,.
$$

Because the $\lambda^* c(L^*)$ and V^* are uniformly bounded, the dominated convergence theorem for $t \to \infty$ gives

$$\mathbb{E}^* \left[\int_0^\infty re^{-rs} h^* \left(\pi a^* - \lambda^* c(L^*) \right) \right] = V^*(W_0) \,.$$

Applying the Itô lemma on $\mathcal{M}$ instead of $\mathcal{M}^*$, I obtain an equation analogous to Equation (13)

$$\begin{aligned}
&\mathbb{E} \left[e^{-rt} V^*(W_t) + \int_0^t re^{-rs} h \left(\pi a - \lambda c(L) \right) ds \right] \\
=& V^*(W_0) + \mathbb{E} \Bigg[\int_0^t re^{-rs - \int_0^s \lambda} \Bigg(- V^* + \pi + (W - u + k) \, (V^*)' \\
&\qquad + \frac{1}{1-L} \frac{k}{0-W} \left(V(0) - V^*(W) - (0 - W)(V^*)' - c(L) \right) \Bigg) \Bigg] \,.
\end{aligned}$$

The integrand is non-positive because it is zero at the maximizer $L^*(W)$.

Because V^* is bounded and the flow monitoring cost is nonnegative, the Fatou lemma for $t \to \infty$ gives

$$V^*(W_0) \geq \mathbb{E} \left[\int_0^\infty re^{-rs} h \left(\pi a - \lambda c(L) \right) ds \right] \,.$$

The same analysis applied to V on $\mathcal{M}$ and then $\mathcal{M}^*$ gives

$$\mathbb{E} \left[\int_0^\infty re^{-rs} h \left(\pi a - \lambda c(L) \right) ds \right] = V(W_0) \geq \mathbb{E}^* \left[\int_0^\infty re^{-rs} h^* \left(\pi a^* - \lambda^* c(L^*) \right) \right] \,.$$

Because $V^*(W_0) \geq V(W_0) \geq V^*(W_0)$, I have $V^*(W_0) = V(W_0)$. Because the FOC (3) admits a unique solution for any $W_0 \in (0, u - k)$, the two incentive schemes $\mathcal{M}$ and $\mathcal{M}^*$ follow in fact the same law.

It remains to verify the discrete-time approachability of $\mathcal{M} = \mathcal{M}^*$. This follows because $V = V_{DL}$ by Lemma 1.

B Proof of Theorem 2

Similar to Theorem 1 in the main model, the proof strategy for Theorem 2 in the extension follows four steps. First, I derive properties of optimal discrete-time incentive schemes. Second, I establish an HJB equation by replicating discrete-time incentive schemes with compound Poisson incentive schemes. Third, I show the optimality of immediate reaction which implies that the continuation value upon Poisson arrival is *either termination or tenure*. This differs from the optimality of termination in the main model because the principal can react to signals by tenuring the agent in addition to terminating him. Fourth, I derive the optimal incentive scheme by *studying the evolution of continuation value absent arrivals*. This differs from the explicit construction in the main model because I can no longer construct the value function with two possible jumps.

For the first three steps, I shall comment on the modifications needed to adapt the lemmata in the main model to the extension. I note here a general modification that applies to all lemmata: the domain of continuation value in the extension is $[0, u]$ instead of $[0, u-k)$ in the main model. For the fourth step of verification, I shall provide a separate proof that takes advantage of the monotonic continuation value.

B.1 Discrete-time incentive provision

B.1.1 Discrete-time problem

The setup of the discrete-time problem is identical to the main model except that the principal needs not incentivize effort when employing the agent, i.e., she may let the agent shirk ($h_n = 1, a_n = 0$).

B.1.2 Simple upper and lower bound of value function

With the extended domain and the possibility of shirking, the first-best value function is $V_{FB}(W) = \frac{W}{u-k}\pi$ for $W \in [0, u-k]$ and $V_{FB}(W) = \frac{u-W}{k}\pi$ for $W \in (u-k, u]$.

Instead of the stationary incentive scheme, I bound $V_{\Delta t}$ from below by 0 noting that the principal can randomize between termination and tenure to provide any continuation value $W \in [0, u]$ and obtain value 0.

B.1.3 Discrete-time value function

In place of Lemma 3, I provide an implicit upper bound to the value function.

Lemma 31 *There exists a concave function $\overline{V}$ satisfying $\overline{V}(W) < V_{FB}(W)$ for $W \in (0, u)$ with $\overline{V}(0) = V_{FB}(0)$ and $\overline{V}(u) = V_{FB}(u)$ such that $V_{\Delta t} \leq \overline{V}$.*

Proof. Analogous to Lemma 3, I prove Lemma 31 by studying a relaxed problem that pools all incentive constraints. The main difference is that the principal can also allow shirking.

1. The agent privately chooses effort $a \in \{0, 1\}$.
2. A public randomization is realized.
3. The principal publicly chooses monitoring $\mathbf{L} \in \Delta_1(0, \infty)$ on the private effort.
4. The principal publicly chooses whether to employ the agent and recommend effort $(h = 1, \tilde{a} = 1)$, employ the agent but not recommend effort $(h = 1, \tilde{a} = 0)$, or not employ the agent $(h = 0, \tilde{a} = 0)$.

The agent's vNM payoff is $h(u - k\tilde{a}a)$, and the principal's $\pi h\tilde{a}a - C(\mathbf{L})$. The problem is parametrized by the agent's continuation value $W_0 \in [0, u]$.

The static incentive scheme follows the same construction except that the recommended effort $\tilde{a}$ equals a_n corresponding to the sequence $\{L_m : m \leq n - 1\}$ in the dynamic scheme. This is because the recommended effort is no longer determined by employment decision h_n.

I compute the agent's expected payoff and show incentive compatibility. For $a = 1$, the agent's expected payoff is

$$\begin{aligned}\mathbb{E}^{a=1}\left[h(u-k\tilde{a}a)\right] =& \sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}\mathbb{E}^{a=1}\left[h_n\left(u-ka_n\right)|n\right] \\ =& \mathbb{E}^{a=1}\left[\sum_{n=1}^{\infty}(1-e^{-r\Delta t})e^{-(n-1)r\Delta t}h(u-ka_n)\right].\end{aligned}$$

Just as in Lemma 3, the payoff in the static incentive scheme equals continuation utility W_0 in the dynamic incentive scheme. Similarly, the static constraint is a particular dynamic incentive constraint and thus is satisfied. The principal's payoff is weakly higher in the static problem, again, due to compound reduction.

I provide an implicit solution to the upper bound because the optimal static scheme optimizes over three instead of two choice variables. The revelation principle implies that it suffices to consider monitoring with ternary support without public randomization. The Inada condition on the monitoring cost implies that an optimum exists and it involves non-trivial monitoring if $W \in (0, u)$. Let $\overline{V}(W)$ denote the value of the static scheme. The non-trivial monitoring is costly and thus $\overline{V}(W) < V_{FB}(W)$ on $(0, u)$. It attains value 0 at $W = 0, u$ because the agent cannot be employed or cannot exert effort. Function $\overline{V}$ is concave by public randomization. ■

The Bellman operators need to incorporate the possibility of shirking. In addition to the working operator and the suspension operator, I define the shirking operator for $v : [0, u] \to \mathbb{R}$ by

$$Sv(W) := e^{-r\Delta t}v\left(e^{r\Delta t}\left(W-(1-e^{-r\Delta t})u\right)\right)$$

where v is taken as $-\infty$ outside its domain. Instead of Equation (7), the maximum operator is defined by

$$Mv(W) := \max\left\{Av(W), Nv(W), Sv(W)\right\}.$$

Lemma 5, which shows a maximizer exists in the closure of the domain $[0, u-k)$, continues to hold in the extension because the domain $[0, u]$ is already compact.

Lemma 7 is no longer needed because of the compact domain $[0, u]$.

I strengthen Lemma 8 which establishes the continuity and concavity of the value function.

Lemma 32 *The value function $V_{\Delta t}$ is continuous and concave with $V_{\Delta t}(0) = V_{\Delta t}(u) = 0$. Moreover, $V'_{\Delta t}(0) \leq \frac{\pi}{u-k}$ and $V'_{\Delta t}(u) \geq -\frac{\pi}{k}$.*

Proof. Concavity and thus continuity in the interior follows from the fixed-point property of the value function. Continuity and the values and marginal values at the boundaries follows from $V_{FB} \geq V_{\Delta t} \geq 0$ and $V_{FB}(0) = V_{FB}(u) = 0$. ■

B.1.4 Optimal discrete-time incentive scheme

Lemma 9, which shows that the randomization operator admits a maximizer, continues to hold but the proof no longer requires the analysis of $J_1 \to u - k$ due to the compact domain.

Lemma 10, which shows that the work operator admits a maximizer in the original domain, is subsumed by the analog of Lemma 5 due to the compact domain.

Recall (lower) cutoff $\underline{W}_{\Delta t} := \max\{W : V_{\Delta t}(W) = V'_{\Delta t}(0)W\}$. I define in addition the upper cutoff $\overline{W}_{\Delta t} := \min\{W : V_{\Delta t}(W) = V'_{\Delta t}(u)(W - u)\}$. Recall also that I call W an extreme point if $(W, V_{\Delta t}(W))$ is an extreme point of the hypograph of $V_{\Delta t}$. Now, I call an extreme point W interior if $W \neq 0, u$.

I adapt Lemma 11 to accommodate the upper cutoff that arises from the possibility of shirking.

Lemma 33 *The cutoffs are bounded by $\underline{W}_{\Delta t} > (1 - e^{-r\Delta t})u$ and $\overline{W}_{\Delta t} < e^{-r\Delta t}u$. Promised utility W is an extreme point if and only if $W = 0, u$ or $W \in [\underline{W}_{\Delta t}, \overline{W}_{\Delta t}]$. For interior extreme point W, work is strictly optimal over suspension and shirking, i.e., $V_{\Delta t}(W) = AV_{\Delta t}(W) > \max\{NV_{\Delta t}(W), SV_{\Delta t}(W)\}$.*

Proof. I prove the lemma by showing that work is strictly optimal at W if it is an interior extreme point, and that W is an interior extreme point for all $W \in [\underline{W}_{\Delta t}, \overline{W}_{\Delta t}]$.

I first show that work is optimal for interior extreme points. Let W be an interior extreme point. The randomization operator R is increasing and so $V_{\Delta t} = RMV_{\Delta t} \geq MV_{\Delta t}$. Because W is an extreme point, I have $V_{\Delta t}(W) = RMV_{\Delta t}(W) = MV_{\Delta t}(W)$. Suppose suspension is optimal $V_{\Delta t}(W) = NV_{\Delta t}(W)$. Because $V_{\Delta t}(0) = 0$, the optimality implies

$$V_{\Delta t}(W) = e^{-r\Delta t} V_{\Delta t}\left(e^{r\Delta t} W\right) = (1 - e^{-r\Delta t}) V_{\Delta t}(0) + e^{-r\Delta t} V_{\Delta t}\left(e^{r\Delta t} W\right)$$

which contradicts with the fact that W is an interior extreme point. Suppose instead shirking is optimal $V_{\Delta t}(W) = SV_{\Delta t}(W)$. Because $V_{\Delta t}(u) = 0$, the optimality implies

$$\begin{aligned} V_{\Delta t}(W) =& e^{-r\Delta t} V_{\Delta t}\left(e^{r\Delta t}\left(W - (1 - e^{-r\Delta t})u\right)\right) \\ =& (1 - e^{-r\Delta t}) V_{\Delta t}(u) + e^{-r\Delta t} V_{\Delta t}\left(e^{r\Delta t}\left(W - (1 - e^{-r\Delta t})u\right)\right) \end{aligned}$$

which contradicts with the fact that W is an interior extreme point. Therefore, $V_{\Delta t}(W) = AV_{\Delta t}(W) > \max\{NV_{\Delta t}(W), SV_{\Delta t}(W)\}$.

I continue to show that W is an interior extreme point for all $W \in [\underline{W}_{\Delta t}, \overline{W}_{\Delta t}]$. By definition, $\underline{W}_{\Delta t}$ is an extreme point. Suppose $\underline{W}_{\Delta t} \leq (1 - e^{-r\Delta t})u$. Then either $\underline{W}_{\Delta t} > 0$ is an interior extreme point or $\underline{W}_{\Delta t} = 0$ and so there exists a sequence of interior extreme points $W_n \to 0$. But working is infeasible and thus suboptimal $AV_{\Delta t}(W) = -\infty$ for $W \in [0, (1 - e^{-r\Delta t})u)$, a contradiction. Therefore, $\underline{W}_{\Delta t} > (1 - e^{-r\Delta t})u$ and so it is an interior extreme point. Analogously, $\overline{W}_{\Delta t}$ is an extreme point by definition. Suppose $\overline{W}_{\Delta t} \geq e^{-r\Delta t}u$. Then either $\overline{W}_{\Delta t} < u$ is an interior extreme point or $\underline{W}_{\Delta t} = 0$ and so there exists a sequence of interior extreme points $W_n \to 0$. But working is infeasible and thus suboptimal $AV_{\Delta t}(W) = -\infty$ for $W \in [e^{-r\Delta t}u, u]$, a contradiction. Therefore, $\overline{W}_{\Delta t} < e^{-r\Delta t}u$ and so it is an interior extreme point.

Therefore, for $W \in (\underline{W}_{\Delta t}, \overline{W}_{\Delta t})$, there exists interior extreme points W_1, W_2 such that $W \in (W_1, W_2)$. Then W is an interior extreme point for all $W \in (W_1, W_2)$ due to the strict concavity of $AV_{\Delta t}$ (Lemma 6) and the monotonicity of R. ■

The definition of the canonical incentive scheme incorporates the possibility of tenure $W = u$ in addition to termination $W = 0$. For $\Delta t > 0$ and $W_0 \in (\underline{W}_{\Delta t}, \overline{W}_{\Delta t})$, I define the canonical discrete-time incentive scheme at continuation value W_0 iteratively. Let $\mathcal{F}_0$ be the trivial measure space. For $n \geq 1$, take an maximizer $\{(p_i, L_i, J_i) : 1 \leq i \leq 4\}$ of $AV_{\Delta t}(W_{n-1})$. Define $(\mathcal{F}_n, \mathbb{P}_n)$ as the product probability space of $(\mathcal{F}_{n-1}, \mathbb{P}_{n-1})$ augmented with n. Define the random variable (L_n, W_n) according to the law of the maximizer (L_i, J_i). Define $h_n := \mathbf{1}_{W_{n-1} \neq 0}$ and $a_n := \mathbf{1}_{W_{n-1} \neq 0, u}$. The complete probability space $(\Omega, \{\mathcal{F}_n\}, \mathbb{P})$ exists by the Kolmogorov extension theorem.

The optimality of the canonical incentive scheme follows from Lemma 33 and the analog of Lemma 12.

B.1.5 Uniform convergence and limit

Lemma 14, which shows the randomization region vanishes as period length shrinks, generalizes to Lemma 34.

Lemma 34 $\lim_{\Delta t \to 0} \underline{W}_{\Delta t} = 0$ *and* $\lim_{\Delta t \to 0} \overline{W}_{\Delta t} = u$.

The proof is symmetric to Lemma 14 and is omitted.

I adapt Lemma 15, which shows the value increases when the period length shrinks, to include the shirking operator and randomization on $(\overline{W}_{\Delta t}, u)$. Note that the statement of the lemma is identical.

Lemma 35 *The value increases when the period length shrinks, i.e.* $\Delta t > \Delta t' > 0$ *implies* $V_{\Delta t'} \geq V_{\Delta t}$.

Proof. Because R and $M_{\Delta t'}$ are increasing operators and $V_{\Delta t'}$ is the fixed point of $RM_{\Delta t'}$, it suffices to show that $RM_{\Delta t'} V_{\Delta t} \geq RM_{\Delta t} V_{\Delta t}$.

The proof of $N_{\Delta t'} V_{\Delta t}(W) \geq N_{\Delta t} V_{\Delta t}(W)$ is identical to Lemma 15. The proof of $S_{\Delta t'} V_{\Delta t}(W) \geq S_{\Delta t} V_{\Delta t}(W)$ is symmetric. The claim is equivalent to

$$(1 - e^{-r\Delta t'}) V_{\Delta t}(u) + e^{-r\Delta t'} V_{\Delta t}(e^{r\Delta t'} \left(W - (1 - e^{-r\Delta t'} u)\right)) \geq (1 - e^{-r\Delta t}) V_{\Delta t}(u) + e^{-r\Delta t} V_{\Delta t}(e^{r\Delta t} \left(W - (\right.$$

because $V_{\Delta t}(u) = 0$. Observe that $(1 - e^{-r\Delta t'})\delta_u + e^{-r\Delta t'} \delta_{e^{r\Delta t'}\left(W - (1 - e^{-r\Delta t'})u\right)}$ is a mean-preserving contraction of $(1 - e^{-r\Delta t})\delta_0 + e^{-r\Delta t} \delta_{e^{r\Delta t}(W - (1 - e^{-r\Delta t} u))}$, where δ is the Dirac delta. Therefore, the claim follows from the concavity of $V_{\Delta t}$.

The proofs of $RM_{\Delta t'}V_{\Delta t}(W) \geq RM_{\Delta t}V_{\Delta t}(W)$ for $W \in [\underline{W}_{\Delta t}, \overline{W}_{\Delta t}]$ and for $W \in [0, \underline{W}_{\Delta t})$ are identical to Lemma 15. It remains to show the claim for $W \in (\overline{W}_{\Delta t}, 0]$. This is symmetric to the lower cutoff. Note that $RM_{\Delta t}V_{\Delta t}(W) = V_{\Delta t}(W) = \frac{W-\overline{W}_{\Delta t}}{u-\overline{W}_{\Delta t}}V_{\Delta t}(u) + \frac{u-W}{u-\overline{W}_{\Delta t}}V_{\Delta t}(\overline{W}_{\Delta t})$. Therefore, I have

$$\begin{aligned} RM_{\Delta t'}V_{\Delta t}(W) \geq & \frac{W-\overline{W}_{\Delta t}}{u-\overline{W}_{\Delta t}}M_{\Delta t'}V_{\Delta t}(u) + \frac{u-W}{u-\overline{W}_{\Delta t}}V_{\Delta t}(\overline{W}_{\Delta t}) \\ \geq & \frac{W-\overline{W}_{\Delta t}}{u-\overline{W}_{\Delta t}}S_{\Delta t'}V_{\Delta t}(u) + \frac{u-W}{u-\overline{W}_{\Delta t}}V_{\Delta t}(\overline{W}_{\Delta t}) \\ \geq & \frac{u-W}{u-\overline{W}_{\Delta t}}V_{\Delta t}(\overline{W}_{\Delta t}) = V_{\Delta t}(W)\,. \end{aligned}$$

The first inequality follows because this is a particular randomization, the second from the definition of $M_{\Delta t'}$ as the maximum, and the third from $S_{\Delta t'}V_{\Delta t}(0) = 0$ and $A_{\Delta t'}V_{\Delta t} \geq A_{\Delta t}V_{\Delta t}$. The last equality follows from the definition of $\overline{W}_{\Delta t}$. ■

B.2 Recursive formulation via Poisson incentive schemes

I adapt Definition 1, which defines compound Poisson incentive schemes, to the possibility of tenure in the canonical incentive scheme. The binary random variable $B \in \{0, 1\}$ distinguishes the end of monitoring due to termination $B = 0$ and due to tenure $B = 1$.

Definition 4 *A tuple* $(\Omega, \mathbb{F}, \mathbb{P}, \Gamma, h, a)$ *is a compound Poisson incentive scheme if there exist discrete-time monitoring technology* $\{\mathbf{L}_n : \mathrm{supp}\,|\mathbf{L}_n| \leq 4, n = 1, 2, ...\}$ *with* $\mathbf{L}_0 := 1$*,* $\mathbf{\Gamma}$*-stopping time* N*,* $\mathbf{\Gamma}_N$*-measurable random variable* B*, and arrival times* $\{\tau_n : n = 0, 1, 2, ...\}$ *with* $\tau_0 := 0$ *of an independent Poisson process of frequency* $\lambda > 0$ *such that*

- *the cumulative excess likelihood ratio is a compound Poisson process*

$$\mathbf{\Gamma}_t = \sum_{n=1}^{N_t \wedge N} (L_n - 1)$$

where $N_t := \inf\{n \geq 0 : \tau_n \leq t\}$*;*

- *the filtration* $(\Omega, \mathbb{F}, \mathbb{P})$ *is the augmented natural filtration of* $(t, \mathbf{\Gamma})$*;*
- *the employment decision is* $h_t = 1$ *iff* $t \leq \tau_N$ *or* $t > \tau_N$ *and* $B = 1$*; the recommended effort is* $a_t = \mathbf{1}_{t \leq \tau_N}$*;*
- *the agent's continuation value* W *satisfies the* ***instantaneous*** *incentive compatibility constraint for* $n \leq N - 1$

$$\lambda \mathbb{E}_{\tau_n} \left[\left(\mathbf{\Gamma}_{\tau_n} - \mathbf{\Gamma}_{\tau_{n+1}} \right) \left(W_{\tau_{n+1}} - W_{\tau_n} \right) \right] = rk \,.$$

Lemma 19, which computes the value and show discrete-time approximation, continues to hold because the principal derives no revenue when the agent is terminated or tenured.

B.2.1 Compound Poisson HJB

The only modification required is the domain $[0, u]$.

B.2.2 Poisson HJB

The only modification required is the domain $[0, u]$.

B.2.3 Smoothness of value function

The only modification required is the domain $[0, u]$.

B.3 Optimality of termination or tenure upon Poisson arrival

B.3.1 Proof of Lemma 36

I adapt Lemma 2, which shows the optimality of termination in discrete time, to the extension and show the optimality of termination or tenure.

Lemma 36 (Immediate reaction) *In any optimal discrete-time incentive schemes, there is a positive-probability signal that leads to either immediate termination or immediate tenure.*

The negation of Lemma 36 is that none of the continuation values is termination or tenure. The proof by contradiction is then identical to Lemma 2 which shows the suboptimality of continuation for all possible signals.

B.3.2 Value function satisfies Termination-Tenure HJB

I adapt Lemma 26, which shows termination minimizes the cost-incentive ratio, to Lemma 37 which shows either termination or tenure minimizes the ratio.

Lemma 37 *For $W \in (0, u)$, let L_0 denote the solution to the first-order condition (3) for termination $J = 0$, and L_u the solution for tenure $J = u$. Then*

$$(L_0, 0) \text{ or } (L_u, u) \in \arg\max_{L,J} \frac{1}{1-L} \frac{k}{J-W} \left(V(J) - V(W) - (J-W)V'(W) - c(L)\right)$$

where the maximization is subject to $(1-L)(J-W) > 0$.

Proof. Lemma 36 implies either (1) there exists a subsequence $\Delta t \to 0$ such that the Δt-optimal incentive scheme all features a signal that leads to immediate termination along the entire subsequence, or (2) there exists a subsequence $\Delta t \to 0$ such that the Δt-optimal incentive scheme all features a signal that leads to immediate tenure along the entire subsequence.

In the first case, the proof of Lemma 26 applies and shows that termination minimizes the cost-incentive ratio. In the second case, the symmetric argument shows that tenure minimizes the ratio. ■

I adapt Proposition 3, which shows the value function satisfies the Termination HJB, to Proposition 4 which shows the value function satisfies the Termination-Tenure HJB.

Proposition 4 (Termination-Tenure HJB) *The value function V is a classical solution to the termination HJB for $W \in (0, u)$*

$$\begin{aligned} v(W) =& \pi + (W - u + k)v'(W) \\ &+ \max \left\{ \max_{L>1} \frac{1}{1-L} \frac{k}{0-W} \left(v(0) - v(W) - (0-W)v'(W) - c(L) \right), \right. \\ &\left. \max_{L<1} \frac{1}{1-L} \frac{k}{u-W} \left(v(u) - v(W) - (u-W)v'(W) - c(L) \right) \right\}. \end{aligned}$$

The proof is almost identical to Proposition 3 except that I invoke Lemma 36 in place of Lemma 26.

B.4 Verification by the evolution of continuation value absent arrivals

B.4.1 Sign and magnitude of drift

When $J = u$ is optimal, I obtain the value v and marginal value v' as functions of L by rewriting the termination-tenure HJB and FOC (3)

$$\begin{cases} v(W) & = \frac{1}{k}\left((u-W)\pi + \left(W - u + k + \frac{k}{L-1} \right) c'(L)(L-1) - (W-u+k)c(L) \right) \\ v'(W) & = \frac{1}{k}\left(-\pi + \left(1 + \frac{1}{1-L}\frac{k}{0-W} \right) c'(L)(L-1) - c(L) \right) \end{cases} \tag{14}$$

in place of Equation (9) when $J = 0$ is optimal. I take derivative of v with respect to W and equating it to v' to obtain a first-order ordinary differential equation of L as a function of W

$$\dot{L} = -\frac{k}{u-W} \frac{c'(L)}{c''(L)} \frac{1}{(L-1)(W-u+k)+k}. \tag{15}$$

in place of Equation (10). I take derivative of v' with respect to W and substituting $\dot{L}$ with the ODE to obtain

$$v''(W) = -\frac{k}{(u-W)^2} c'(L) \frac{1}{(W-u+k) + \frac{k}{L-1}}. \tag{16}$$

in place of Equation (11). For the value function to be strictly concave, the drift $r\left((W-u+k) + \frac{k}{L-1}\right)$ must be negative. As a result, $\dot{L} < 0$.

With abuse of notation, I write termination as a short hand for monitoring with Poisson bad news that leads to immediate termination, and tenure short for monitoring with Poisson good news that leads to immediate tenure.

Lemma 27 shows that the solution to ODE (10) is well-behaved for termination. I also need to establish Lemma 38 which shows the solution to ODE (15) is well-behaved for tenure.

Lemma 38 *Consider the initial value problem of ODE (15) at* $(\tilde{W}, \tilde{L})$ *on the domain* $W \in [0, u)$ *and* $L \in (0, 1)$*. It admits a unique solution on* $[\tilde{W}, u)$*.*

Proof. Note that the domain satisfies $W - u + k + \frac{k}{L-1} < 0$.

The RHS of Equation (15) satisfies the Picard-Lindelöf conditions, i.e. continuous in W and Lipschitz continuous in L on all compact subsets of the domain, so there exists a unique solution in the interior. Moreover, $\dot{L} \in (0, \infty)$ so the solution also solves the initial value problem

$$\dot{W} = \dot{L}^{-1} = -\frac{u-W}{k} \frac{c''(L)}{c'(L)} \left((L-1)(W-u+k) + k\right) \tag{17}$$

with the same initial condition and domain.

Define L_0 as a function of $W \in (0, u)$ as the unique solution to the first-order condition (3) with $J = 0$. Let $\lambda_0 := \frac{1}{1-L_0} \frac{rk}{0-W} > 0$ denote the corresponding frequency for binding IC. Similarly, L_u is the unique solution with $J = u$ and $\lambda_u := \frac{1}{1-L_u} \frac{rk}{u-W} > 0$ denotes the corresponding frequency. ∎

Lemma 39 *There exists uniform bound* $\delta > 0$ *such that the drift from termination* $W - u + k + \frac{k}{L_0(W)-1} > \delta$ *for all* $W \in (0, u)$*.*

Proof. Because the drift is nonnegative by the concavity of value function (Equation (11)), it suffices to show that $L_0(\cdot)$ lies in a compact region strictly bounded away from the curve $W - u + k + \frac{k}{L(W)-1} = 0$.

The region is defined by four constraints $W \in [0, u]$, $L \geq 1$, $L(W) \leq L^*(W)$, and $L \leq L_{\max}$ where L^* is the optimal likelihood ratio function in the main model, and $L_{\max}$ is the unique solution on $(1, \infty)$ to

$$c'(L)(L-1) - c(L) = \pi + \pi \max\left\{\frac{u}{u-k}, \frac{u}{k}\right\}.$$

The first constraint is on the domain of continuation value, and the second constraint follows from the IC constraint that bad news $L > 1$ follows termination $J = 0 < W$. The third constraint provides a variable upper bound on the optimal likelihood ratio. Observe from Equation (9) that $v'(W)$ is decreasing in L, and thus $v(W)$ is also decreasing in L. Because the optimal incentive scheme in the main model is feasible in the extension, the value in the extension is weakly higher and therefore the optimal likelihood ratio is weakly lower. The fourth constraint provides a uninform upper bound on the optimal likelihood ratio. It follows from the first-order condition and the fact that the value function and its derivative are both uniformly bounded. ■

Lemma 40 *There exists uniform bound $\delta > 0$ such that the drift from tenure $W - u + k + \frac{k}{L_0(W)-1} < -\delta$ for all $W \in (0, u)$.*

Proof. Because the drift is nonnegative by the concavity of value function (Equation (16)), it suffices to show that $L_u(\cdot)$ lies in a compact region strictly bounded away from the curve $W - u + k + \frac{k}{L(W)-1} = 0$.

The region is defined by $W \in [0, u]$, $L \leq 1$, and $L \geq L_{\min}$ where $L_{\min}$ is the unique solution on $(0, 1)$ to

$$c'(L)(L-1) - c(L) = \pi + \pi \max\left\{\frac{u}{u-k}, \frac{u}{k}\right\}.$$

The first constraint is on the domain of continuation value, and the second constraint follows from the IC constraint that good news $L > 1$ follows tenure $J = u < W$. The third constraint provides a uninform upper bound on the optimal likelihood ratio.

It follows from the first-order condition and the fact that the value function and its derivative are both uniformly bounded. ■

B.4.2 Four possible forms

Proof of Theorem 2. Let $W_0 \in \arg\max_W V(W)$ denote the maximizer of the value function. It is unique and it is in the interior $W_0 \in (0, u)$ because V is strictly concave and $V(0) = V(u) = 0$. I show that the maximum value $V(W_0)$ is attained by constructing incentive schemes of the four possible forms.

For $W \in (0, u)$ such that termination and tenure are both optimal, define function α by

$$\alpha(W) := \frac{-\left(W - u + k + \frac{k}{L_u(W)-1}\right)}{\left(W - u + k + \frac{k}{L_0(W)-1}\right) - \left(W - u + k + \frac{k}{L_u(W)-1}\right)}$$

such that the drift is zero such that termination at frequency $\alpha\lambda_0$ and tenure at frequency $(1-\alpha)\lambda_u$ yields zero drift for the continuation value

$$r\left(W - u + k - \alpha\lambda_0(0 - W) - (1-\alpha)\lambda_u(u - W)\right) = 0\,.$$

The function is a fraction $\alpha(W) \in (0, 1)$ due to Lemma 39 and Lemma 40.

I consider five cases categorized by the optimality of termination and/or tenure. The first case is that termination and tenure both are optimal at W_0. The incentive scheme is a special case of the third and fourth form in that the trial period is degenerate. The monitoring technology Γ is the sum of two stationary compensated Poisson processes, the Poisson bad news with jump $L_0(W_0) - 1$ at frequency $\alpha(W_0)\lambda_0(W_0)$ and the Poisson good news with jump $L_u(W_0) - 1$ at frequency $(1 - \alpha(W_0))\lambda_u(W_0)$, stopped at the first arrival τ. Bad news leads to immediate termination $W = 0$ and good news leads to immediate tenure $W = u$.

I show that this incentive scheme gives the agent continuation value $W_t = W_0$ conditional on no news. For $t < \tau$ and $s > 0$, the continuation value at t up to s

satisfies

$$
\begin{aligned}
&\mathbb{E}_t\left[\int_t^{s\wedge\tau} re^{-r(q-t)}(u-k)dq + e^{-r(s\wedge\tau-t)}W_{s\wedge\tau}\right] \\
=&W_t + \mathbb{E}_t\left[\int_t^{s\wedge\tau} e^{-r(q-t)}\left(-rW + r(u-k) + \alpha\lambda_0(0-W) + (1-\alpha)\lambda_u(u-W)\right)dq\right] \\
=&W_t = W_0\,.
\end{aligned}
$$

The first equality follows from the Itô lemma. The second equality follows from the definition of α. Because s is arbitrary, I take the limit $s\to\infty$ to obtain the continuation value at t

$$
\mathbb{E}_t\left[\int_t^{\tau} re^{-r(q-t)}(u-k)dq\right] = \lim_{s\to\infty}\mathbb{E}_t\left[\int_t^{s\wedge\tau} re^{-r(q-t)}(u-k)dq + e^{-r(s\wedge\tau-t)}W_{s\wedge\tau}\right] = W_0
$$

where the first equality follows from uniform integrability.

I show that this incentive scheme is incentive compatible. For ease of exposition, I only consider a' that does not exert effort when the agent is tenured. It suffices to show that the agent's continuation value is independent of a'. For any such a' and $t>0$, the continuation value at time 0 up to t is independent of a'

$$
\begin{aligned}
&\mathbb{E}^{a'}\left[\int_0^{t\wedge\tau} re^{-rs}(u-ka')ds + e^{-r(t\wedge\tau)}W_{t\wedge\tau}\right] \\
=&W_0 + \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-rs}\Big(-rW + r(u-ka') + \alpha\lambda_0L_0^{1-a'}(0-W) + (1-\alpha)\lambda_uL_u^{1-a'}(u-W)\Big)ds\right] \\
=&W_0 + \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-rs}\left(-rW + r(u-k) + \alpha\lambda_0(0-W) + (1-\alpha)\lambda_u(u-W)\right)ds\right] \\
=&W_0\,.
\end{aligned}
$$

The first equality follows from the Itô lemma and the change of measure from $\mathbb{P}^{a'}$ to $\mathbb{P}$. The second equality follows from the instantaneous IC. The third equality follows from the definition of α. Because t is arbitrary, I take $t\to\infty$ to obtain the continuation value of a'

$$\mathbb{E}^{a'}\left[\int_0^\tau re^{-rt}(u-ka')dt\right]=\lim_{t\to\infty}\mathbb{E}^{a'}\left[\int_0^{t\wedge\tau}re^{-rs}(u-ka')ds+e^{t\wedge\tau}W_{t\wedge\tau}\right]=W_0$$

where the first equality follows from uniform integrability.

I show that this incentive scheme attains value $V(W_0)$. It can therefore be approximated by discrete-time incentive schemes because $V(W_0)=V_{DL}(W_0)$ by Lemma 20. For $t>0$, the principal's value up to time t is

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-rs}\left(r\pi ds-dC_s(\Gamma)\right)+e^{-rt\wedge\tau}V(W_{t\wedge\tau})\right]\\
=&\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-rs}\left(r\pi-\alpha\lambda_0c(L_0)-(1-\alpha)\lambda_uc(L_u)\right)ds+e^{-rt\wedge\tau}V(W_{t\wedge\tau})\right]\\
=&V(W_0)+\mathbb{E}\Bigg[\int_0^{t\wedge\tau}e^{-rs}\Bigg(-rV(W_s)+r\pi-\alpha\lambda_0\left(V(W_s)+c(L_0)\right)\\
&\qquad\qquad-(1-\alpha)\lambda_u\left(V(W_s)+c(L_u)\right)\Bigg)ds\Bigg]\\
=&V(W_0)\\
&+\mathbb{E}\Bigg[\int_0^{t\wedge\tau}e^{-rs}\Bigg(-rV+r\pi+r(W-u+k)V'-\alpha\lambda_0\left(V(0)-V-(0-W)V'-c(L_0)\right)\\
&\qquad\qquad-(1-\alpha)\lambda_u\left(V(u)-V-(u-W)V'-c(L_u)\right)\Bigg)ds\Bigg]\\
=&V(W_0)\,.
\end{aligned}$$

The first equality follows from the monitoring cost for jump processes. The second equality follows from the Itô lemma. The third equality follows from algebra. The fourth equality follows from the HJB equation because termination and tenure are both optimal. Because t is arbitrary, I take $t\to\infty$ to obtain the principal's value

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\tau e^{-rs}\left(r\pi ds-dC_s(\Gamma)\right)+e^{-r\tau}V(W_\tau)\right]\\
=&\lim_{t\to\infty}\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-rs}\left(r\pi ds-dC_s(\Gamma)\right)+e^{-rt\wedge\tau}V(W_{t\wedge\tau})\right]\\
=&V(W_0)\,.
\end{aligned}$$

The first equality follows from uniform integrability.

The second case is that termination is strictly optimal on $[W_0, u)$. The incentive scheme corresponds to the first form in Theorem 2. Recall $L_0(W)$ is the unique solution to the first-order condition (3) for $J = 0$, and $\lambda_0(W)$ is the corresponding frequency. Let W_t^* denote the solution to $\frac{dW_t}{dt} = r\left(W_t - u + k + \frac{k}{1-L_0(W_t)}\right)$ with initial condition $W_0^* = W_0$. Let $T := \inf\{t : W_t^* = u\}$ denote the time at which W^* reaches u. It is finite because the derivative of W^* is bounded away from zero by Lemma 39. The monitoring technology Γ is the non-stationary compensated Poisson process with jump $L_0(W_t^*) - 1$ at frequency $\lambda_0(W_t^*)$, stopped at the first arrival τ and at T whichever is earlier. Bad news leads to immediate termination and no arrival before T leads to tenure.

I show that the agent's continuation value at time $t \in [0, T]$ is W_t^*.

$$
\begin{aligned}
&\mathbb{E}_t\left[\int_t^{T\wedge\tau} re^{-r(s-t)}(u-k)ds + e^{-r(T\wedge\tau - t)}W_{T\wedge\tau}\right] \\
=&W_t^* + \mathbb{E}_t\left[\int_t^{T\wedge\tau} e^{-r(s-t)}\left(-rW^* + r(u-k) + r\left(W^* - u + k + \frac{k}{1-L_0}\right) + \lambda_0(0 - W^*)\right)ds\right] \\
=&W_t^*\,.
\end{aligned}
$$

The first equality follows from the Itô lemma and the second equality from the definition of L_0 and λ_0.

I show incentive compatibility. For ease of exposition, I only consider a' that does not exert effort when the agent is tenured. It suffices to show that the agent's continuation value is independent of a'

$$
\begin{aligned}
&\mathbb{E}^{a'}\left[\int_0^{T\wedge\tau} re^{-rt}(u-ka')dt + e^{-rT\wedge\tau}W_{T\wedge\tau}\right] \\
=&W_0^* + \mathbb{E}\left[\int_0^{T\wedge\tau} e^{-rt}\left(-rW^* + r(u-ka') + \left(W^* - u + k + \frac{k}{1-L_0}\right) + \lambda_0 L_0^{1-a'}(0 - W^*)\right)ds\right] \\
=&W_0^* + \mathbb{E}\left[\int_0^{T\wedge\tau} e^{-rt}\left(-rW^* + r(u-k) + \left(W^* - u + k + \frac{k}{1-L_0}\right) + \lambda_0(0 - W^*)\right)ds\right] \\
=&W_0^*\,.
\end{aligned}
$$

The first equality follows from the Itô lemma and the change of measure from $\mathbb{P}^{a'}$ to $\mathbb{P}$. The second equality follows from the instantaneous incentive constraint. The third equality follows from the definitions of L_0 and λ_0.

I show the incentive scheme attains value $V(W_0)$. It can therefore be approximated by discrete-time incentive schemes because $V(W_0) = V_{DL}(W_0)$ by Lemma 20. The principal's value is

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T\wedge\tau} e^{-rt}\left(r\pi dt - dC_t(\Gamma)\right) + e^{rT\wedge\tau}V(W_{T\wedge\tau})\right] \\
=&\mathbb{E}\left[\int_0^{T\wedge\tau} e^{-rt}\left(r\pi - \lambda_0 c(L_0)\right) ds + e^{rT\wedge\tau}V(W_{T\wedge\tau})\right] \\
=&V(W_0) + \mathbb{E}\left[\int_0^{T\wedge\tau} e^{-rt}\left(r\pi + r(W-u+k)V' + \lambda_0\left(V(0) - V - (0-W)V' - c(L_0)\right)\right) ds\right] \\
=&V(W_0)\,.
\end{aligned}
$$

The first equality follows from the monitoring cost for jump processes. The second equality follows from the Itô lemma. The third equality follows from the HJB equation by the optimality of termination.

The third case is that there exists $\tilde{W} \in (W_0, u)$ such that termination is strictly optimal on $[W_0, \tilde{W}]$ and termination and tenure are both optimal at $\tilde{W}$. The optimal incentive scheme corresponds to the third form in Theorem 2. Let W_t^* denote the solution to $\frac{dW_t}{dt} = r\left(W_t - u + k + \frac{k}{1-L_0(W_t)}\right)$ with initial condition $W_0^* = W_0$. Let $T := \inf\{t : W_t^* = u\}$ denote the time at which W^* reaches $\tilde{W}$. It is finite because the derivative of W^* is bounded away from zero by Lemma 39. The monitoring technology Γ consists of the non-stationary Poisson bad news monitoring during trial period $[0,T]$ and the stationary two-sided Poisson monitoring afterwards (T,∞), stopped at the first arrival τ. During the trial period, it is the non-stationary compensated Poisson process with jump $L_0(W_t^*) - 1$ at frequency $\lambda_0(W_t^*)$. Bad news leads to immediate termination $W = 0$. After the trial period, it is the sum of two stationary compensated Poisson processes, the Poisson bad news with jump $L_0(\tilde{W}) - 1$ at frequency $\alpha(\tilde{W})\lambda_0(\tilde{W})$ and the Poisson good news with jump $L_u(\tilde{W}) - 1$ at frequency $(1-\alpha(\tilde{W}))\lambda_u(\tilde{W})$. Bad news leads to immediate termination $W = 0$ and good news leads to immediate tenure $W = u$.

I show that the agent's continuation value at time t is W_t^*. For $t \in [0,T]$ and $s > T$, the agent's continuation value at time t up to s is

$$
\begin{aligned}
&\mathbb{E}_t\left[\int_t^{s\wedge\tau} re^{-r(q-t)}(u-k)dq + e^{-r(q\wedge\tau-t)}W_{q\wedge\tau}\right] \\
=&W_t^* \\
&+\mathbb{E}_t\Bigg[\int_t^{T\wedge\tau} e^{-r(s-t)}\left(-rW^* + r(u-k) + r\left(W^* - u + k + \frac{k}{1-L_0}\right) + \lambda_0(0-W^*)\right)ds \\
&\qquad + \int_{T\wedge\tau}^{s\wedge\tau} e^{-r(s-t)}\left(-rW^* + r(u-k) + \alpha\lambda_0(0-W^*) + (1-\alpha)\lambda_u(u-W^*)\right)ds\Bigg] \\
=&W_t^*\,.
\end{aligned}
$$

The first equality follows from the Itô lemma. The second equality follows from the definitions of L_0, λ_0, and α. Because s is arbitrary, I take $s \to \infty$ to obtain the agent's continuation value at t

$$
\mathbb{E}_t\left[\int_t^{\tau} re^{-r(s-t)}(u-k)ds\right] = \lim_{s\to\infty}\mathbb{E}_t\left[\int_t^{s\wedge\tau} re^{-r(q-t)}(u-k)dq + e^{-r(q\wedge\tau-t)}W_{q\wedge\tau}\right] = W_t^*\,.
$$

The first equality follows from uniform integrability. For $t > T$, the agent's continuation value is $W_t^* = \tilde{W}$. The derivation is identical to that of the first case with W_0 replaced by $\tilde{W}$ because the continuation of the incentive scheme is the same.

I show incentive compatibility. For ease of exposition, I only consider a' that does not exert effort when the agent is tenured. It suffices to show that the agent's continuation value is independent of a'. For $t > T$, the agent's continuation value of

a' at 0 up to t is

$$\begin{aligned}
&\mathbb{E}^{a'}\left[\int_0^{t\wedge\tau} re^{-rs}(u-ka')ds + e^{-rt\wedge\tau}W_{t\wedge\tau}\right] \\
=&W_0^* \\
&+\mathbb{E}\Bigg[\int_0^{T\wedge\tau} e^{-rs}\left(-rW^* + r(u-ka') + r\left(W^* - u + k + \frac{k}{1-L_0}\right) + \lambda_0 L_0^{1-a'}(0-W^*)\right)ds \\
&\qquad + \int_{T\wedge\tau}^{t\wedge\tau} e^{-rs}\left(-rW^* + r(u-ka') + \alpha\lambda_0 L_0^{1-a'}(0-W^*) + (1-\alpha)\lambda_u L_u^{1-a'}(u-W^*)\right)ds\Bigg] \\
=&W_0^* \\
&+\mathbb{E}\Bigg[\int_0^{T\wedge\tau} e^{-rs}\left(-rW^* + r(u-k) + r\left(W^* - u + k + \frac{k}{1-L_0}\right) + \lambda_0(0-W^*)\right)ds \\
&\qquad + \int_{T\wedge\tau}^{t\wedge\tau} e^{-rs}\left(-rW^* + r(u-k) + \alpha\lambda_0(0-W^*) + (1-\alpha)\lambda_u(u-W^*)\right)ds\Bigg] \\
=&W_0^*\,.
\end{aligned}$$

The first equality follows from the Itô lemma and the change of measure from $\mathbb{P}^{a'}$ to $\mathbb{P}$. The second equality follows from the instantaneous incentive constraint. The third equality follows from the definitions of L_0, λ_0, and α. Because t is arbitrary, I take $t \to \infty$ to obtain the agent's continuation of a'

$$\mathbb{E}^{a'}\left[\int_0^{\tau} re^{-rt}(u-ka')ds\right] = \lim_{t\to\infty}\mathbb{E}^{a'}\left[\int_0^{t\wedge\tau} re^{-rs}(u-ka')ds + e^{-rt\wedge\tau}W_{t\wedge\tau}\right] = W_0^*\,.$$

The first equality follows from uniform integrability.

I show the incentive scheme attains value $V(W_0)$. It can therefore be approximated by discrete-time incentive schemes because $V(W_0) = V_{DL}(W_0)$ by Lemma 20. For

$t > T$, the principal's value up to time t is

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge\tau} e^{-rs}\left(r\pi ds - dC_s(\Gamma)\right) + e^{rt\wedge\tau}V(W_{t\wedge\tau})\right] \\
=&\mathbb{E}\left[\int_0^{T\wedge\tau} e^{-rs}\left(r\pi ds - dC_s(\Gamma)\right) + \int_{T\wedge\tau}^{t\wedge\tau} e^{-rs}\left(r\pi ds - dC_s(\Gamma)\right) + e^{rt\wedge\tau}V(W_{t\wedge\tau})\right] \\
=&\mathbb{E}\Bigg[\int_0^{T\wedge\tau} e^{-rs}\left(r\pi - \lambda_0 c(L_0)\right) ds + \int_{T\wedge\tau}^{t\wedge\tau} e^{-rs}\left(r\pi - (\alpha\lambda_0 c(L_0) + (1-\alpha)\lambda_u c(L_u))\right) ds \\
&\quad + e^{rt\wedge\tau}V(W_{t\wedge\tau})\Bigg] \\
=&V(W_0) \\
&+\mathbb{E}\Bigg[\int_0^{T\wedge\tau} e^{-rt}\left(r\pi + r(W-u+k)V' + \lambda_0\left(V(0) - V - (0-W)V' - c(L_0)\right)\right) ds \\
&\quad + \int_{T\wedge\tau}^{t\wedge\tau} e^{-rs}\Big(- rV + r\pi + r(W-u+k)V' - \alpha\lambda_0\left(V(0) - V - (0-W)V' - c(L_0)\right) \\
&\qquad\qquad - (1-\alpha)\lambda_u\left(V(u) - V - (u-W)V' - c(L_u)\right)\Big) ds\Bigg] \\
=&V(W_0)\,.
\end{aligned}$$

The second equality follows from the monitoring cost for jump processes. The third equality follows from the Itô lemma. The fourth equality follows from the HJB equation by the optimality of termination during the trial period when $W_t^* \in [W_0, \tilde{W}]$ and the optimality of termination and tenure afterwards when $W_t^* = \tilde{W}$. Because t is arbitrary, I take $t \to \infty$ to obtain the principal's value

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau} e^{-rs}\left(r\pi ds - dC_s(\Gamma)\right) + e^{-r\tau}V(W_\tau)\right] \\
=&\lim_{t\to\infty}\mathbb{E}\left[\int_0^{t\wedge\tau} e^{-rs}\left(r\pi ds - dC_s(\Gamma)\right) + e^{-rt\wedge\tau}V(W_{t\wedge\tau})\right] \\
=&V(W_0)\,.
\end{aligned}$$

The first equality follows from uniform integrability.

The fourth case is that tenure is strictly optimal on $(0, W_0]$. The incentive scheme corresponds to the second form in Theorem 2. The construction and verification are symmetric to the second case and thus omitted.

The fifth case is that there exists $\tilde{W} \in (0, W_0)$ such that tenure is strictly optimal on $[\tilde{W}, W_0]$ and both termination and tenure are optimal at $\tilde{W}$. The optimal incentive scheme corresponds to the fourth form in Theorem 2. The construction and verification are symmetric to the third case and thus omitted. ■